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Elements of ANALYTIC GEOMETRY

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PREFACE

The first edition of the *Elements* was an abridgment of the author's *Analytic Geometry*, second edition. The present book has been prepared, not by revising the first edition, but by abridging the third (1938) edition of the longer book. It is an abridgment in the accurate sense, not a condensation — that is, brevity has been obtained not by skimping explanations or leaving out worked examples, but by omitting bodily various topics of less vital importance.

The book differs from its predecessor as follows:

Articles 28–29 and 44, on linear and quadratic functions, are new. The focal-distance definition of the central conics is introduced early (§§ 46, 52). The chapter on properties of the conics is considerably abbreviated. Otherwise the treatment of first- and second-degree loci is not greatly changed.

Chapter X, discussing rather thoroughly the graphs of rational algebraic functions, is new. This is followed by the chapter on polar coördinates. The section devoted to plane geometry concludes with a short chapter on parametric representation.

The treatment of solid geometry has not been greatly changed, except that more effective arrangement in the early stages should result in some saving of time in the classroom.

The drill exercises have been almost entirely worked over. A number of theorems that might appear in the text of a longer book have been incorporated in the exercises, to be developed or not as the teacher prefers, but available for reference in any case. See, for instance, Ex. 24, p. 58, or Ex. 11, p. 99.

Inevitably, some instructors will fail to find a presentation of certain topics that they would like to teach. For example, within the space allotted it was impossible to incorporate a chapter on curve-fitting. It is hoped that such instructors may find the author's longer book suited to their purpose.

CLYDE E. LOVE

ANN ARBOR, MICHIGAN
April, 1940

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Elements of ANALYTIC GEOMETRY

PLANE ANALYTIC GEOMETRY

CHAPTER I

CARTESIAN COÖRDINATES

1. Directed line segments. When a line segment is traced in a definite sense *from* one endpoint *to* the other, the segment is said to be *directed*. If the terminal points are A, B, we speak of the segment AB or the segment BA according as the sense is from A to B or from B to A.

If one sense is chosen as positive, then the opposite sense is negative: thus

$$AB = -BA, \qquad \text{or} \qquad AB + BA = 0.$$

If C is any third point of the straight line through A and B, then for all possible positions of A, B, and C we have

$$AB + BC = AC,$$

or

$$AB + BC + CA = 0.$$

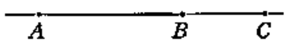

FIG. 1

Two directed line segments lying in the same line or in parallel lines are said to be *equal* if they have the same length and are measured in the same sense.

2. Position of a point in a plane. If a point lies in a given plane, two magnitudes, or "coördinates," are necessary to determine its position, each coördinate being measured in a definite sense. Thus the position of a picture on a wall may be given by its distance to the right (or left) of a window and its height above the floor.

3. Cartesian coördinates. Given a point P lying in a certain plane, let us assume two perpendicular lines Ox, Oy lying in the plane. The line Ox is called the *x-axis*, Oy the *y-axis*, and their point of intersection O is the *origin*. The position of P is evidently known if its distances from the axes are given, each being measured in a definite sense, *from* the axis *to* the point. These directed segments are called the *Cartesian coördinates* of P: the distance from the y-axis — NP or its equal OM — is the *abscissa*, the distance MP from the x-axis is the *ordinate*. Each coördinate is represented algebraically by a *number*.

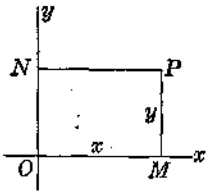

FIG. 2

We shall ordinarily assume the axes as in Fig. 2, and shall consider abscissas *positive if measured to the right, negative if measured to the left*; ordinates *positive if measured upward, negative if measured downward.*

It is customary to write the coördinates of a point in parentheses, with the abscissa first: thus in Fig. 3 the point $P: (3, 5)$, also written simply $(3, 5)$, has the abscissa 3 and the ordinate 5. The figure also shows the points $Q: (2, -4)$, $R: (-4, -3)$, and $S: (-3, 0)$.

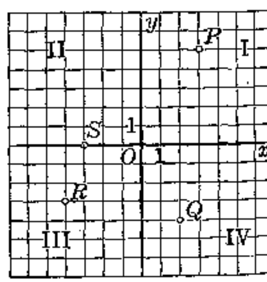

FIG. 3

The axes divide the plane into four compartments, called *quadrants*, and numbered as in Fig. 3. The abscissa is positive in the first and fourth quadrants, the ordinate positive in the first and second.

In all work with Cartesian coördinates, it will in general be assumed that line segments oblique to the axes are *undirected*, segments parallel to an axis are *directed*. Segments parallel to Ox will be considered positive if measured to the right, negative to the left; segments parallel to Oy, positive upward, negative downward.

By the introduction of a Cartesian coördinate system there is set up a unique correspondence between *points*, on the one hand, and *pairs of real numbers*, on the other: *to every (ordered) pair of real numbers there corresponds one and only one point in the plane, and conversely.**

4. Units. In analytic geometry, drawings involving Cartesian coördinates are usually made on square-ruled paper, called *coördinate paper* (see Fig. 3). The unit of measurement chosen need not, and usually should not, be the width of one space on the coördinate paper; the scale should be selected with regard to the nature of the drawing to be made — neither so large that some of the points fall beyond the limits of the paper, nor so small that the properties of the figure become obscured. The scale adopted should be clearly indicated.

Cases sometimes arise in which it is convenient to adopt different scales on the two axes, which of course produces a distortion of the figure (see Exs. 37–40 below). Except where the contrary is stated we shall assume always that the unit for ordinates is the same as that for abscissas.

To plot a point whose coördinates are irrational, we employ decimal approximations. For instance, to plot the point $(\sqrt{2}, \sqrt{3})$, we might take † $\sqrt{2} = 1.41$, $\sqrt{3} = 1.73$.

* The coördinates defined above are more precisely called *rectangular* Cartesian coördinates. It is possible to set up a system in which the axes are oblique to each other, but in this book only the rectangular system will be used.

The word *Cartesian* is derived from the name of *René Descartes* (1596–1650), Latinized as *Cartesius*, who was the founder of analytic geometry.

† Of course $(\sqrt{2}, \sqrt{3})$ and (1.41, 1.73) are not at all the same point — we are merely doing the best we can for plotting purposes. Such approximations are permissible only in plotting, which is an approximate process at best, or in applications where an approximate result is satisfactory.

5. Distance between two points. The distance between two points P_1, P_2 can be expressed in terms of their coördinates by the Theorem of Pythagoras. Let the coördinates of the two points be denoted by the letters x, y with subscripts: $P_1 : (x_1, y_1)$, $P_2 : (x_2, y_2)$. Now, in Fig. 4,

(1) $$d = \sqrt{\overline{P_1Q}^2 + \overline{QP_2}^2}.$$

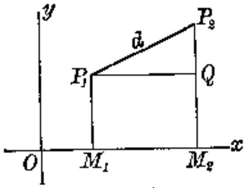

FIG. 4

But

$$P_1Q = M_1M_2 = OM_2 - OM_1,$$
$$QP_2 = M_2P_2 - M_2Q:$$

that is,

$$P_1Q = x_2 - x_1,$$
$$QP_2 = y_2 - y_1.$$

Substituting in (1), we find

(2) $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

By drawing the figure in various positions, the student may convince himself that the formula holds no matter where the points P_1, P_2 may be situated — not merely when both points lie in the first quadrant.

Examples: (*a*) Find the distance between the points (3, 2) and (− 5, 4).

By formula (2), or directly from a figure, we find

$$d = \sqrt{(-8)^2 + 2^2} = \sqrt{68} = 2\sqrt{17}.$$

(*b*) A point moves so that its distance from the origin is always 2. Express this fact by an algebraic equation.

Let the coördinates of the moving point be (x, y). Then, by (2), the distance of this point from the origin is $\sqrt{x^2 + y^2}$. Hence the required equation is

$$\sqrt{x^2 + y^2} = 2, \qquad \text{or} \qquad x^2 + y^2 = 4.$$

EXERCISES

Draw the following figures on coördinate paper, choosing a suitable scale in each instance.

1. Triangle with vertices (1, 3), (5, 0), (− 2, − 1).

2. Quadrilateral with vertices (5, − 2), (4, 3), (− 2, 3), (2, − 5).

3. Quadrilateral with vertices $(\frac{1}{2}, -\frac{7}{8})$, $(\frac{5}{8}, 0)$, $(-\frac{1}{4}, 0)$, $(-\frac{1}{2}, -\frac{1}{8})$.

4. Triangle with vertices (24, 12), (− 30, 6), (15, − 15).

5. What can be said of the coördinates of all points on the x-axis? On the y-axis? On the line through O bisecting the first and third quadrants? The second and fourth quadrants? On the line parallel to the y-axis two units to the right of it? Two units to the left?

6. Where does a point lie if its abscissa is 0? If its ordinate is 0? If abscissa and ordinate are equal? Are numerically equal but of opposite sign?

Find the distance between the given points.

7. (5, 3), (6, 7). *Ans.* $\sqrt{17}$.

8. (− 6, 2), (− 4, − 3). *Ans.* $\sqrt{29}$.

9. $(\frac{3}{2}, -\frac{7}{4})$, $(-\frac{9}{4}, -\frac{5}{2})$. *Ans.* $\frac{3}{4}\sqrt{10}$.

10. $(\frac{5}{6}, -\frac{1}{2})$, $(\frac{1}{4}, -\frac{4}{3})$. *Ans.* $\frac{1}{12}\sqrt{149}$.

11. Prove that the points (5, 0), (2, 1), (4, 7) are the vertices of a right triangle, and find its area. *Ans.* $A = 10$.

12. Prove that the points (− 2, 1), (0, − 5), (10, 5) are the vertices of a right triangle, and find its area. *Ans.* $A = 40$.

13. Prove that the points (3, 2), (7, − 2), (6, 1) are the vertices of an isosceles triangle, and find its area. *Ans.* $A = 4$.

14. Prove that the points (1, 3), (3, − 1), (7, − 3) are the vertices of an isosceles triangle, and find its area. *Ans.* $A = 6$.

15. Prove that the points (− 7, 1), (5, − 4), (10, 8), (− 2, 13) are the vertices of a square, and find its area. *Ans.* $A = 169$.

16. Prove that the points $(-\frac{3}{2}, 4)$, $(-\frac{7}{2}, 3)$, $(-\frac{3}{2}, -1)$, $(\frac{1}{2}, 0)$ are the vertices of a parallelogram. Is the parallelogram a rectangle?

17. Draw the circle with center at (3, 2) passing through (13, − 10). Does this circle pass through (− 11, 9)?

18. Draw a circle with center at (0, − 13) tangent to the x-axis. Does this circle pass through (11, − 6)? Through (− 5, − 1)?

19. Find the radius of a circle with center at (3, 1), if a chord of length 6 is bisected at (6, 5). *Ans.* $\sqrt{34}$.

20. Find the radius of a circle with center at (1, − 1), if a chord of length 10 is bisected at (2, 0).

21. The center of a circle is at (5, 3) and its radius is 5. Find the length of the chord that is bisected at (3, 2). *Ans.* $4\sqrt{5}$.

22. The center of a circle is at (6, − 1) and its radius is 6. Find the length of the chord that is bisected at (3, 4). *Ans.* $2\sqrt{2}$.

23. Prove that the quadrilateral with vertices (0, 4), (7, − 7), (2, − 2), (1, − 9) consists of two equal triangles placed base to base, and find its area. *Ans.* $A = 20$.

24. Prove that the lines joining (6, 2), (13, 1), (12, − 6), (1, − 8) form a kite-shaped quadrilateral, and find its area. *Ans.* $A = 75$.

By division into triangles in a suitable way, find the area of the quadrilateral having the given points as vertices.

25. (8, 4), (7, − 3), (− 4, − 5), (1, 5). *Ans.* 75.

26. (3, 2), (− 8, 9), (− 3, 4), (− 10, 3). *Ans.* 20.

27. (0, 0), (− 3, 4), (− 7, 1), (− 6, − 6).

28. (− 1, 3), (0, 5), (− 7, − 4), (9, − 2).

Determine whether the given points lie in a straight line.

29. (− 2, − 8), (1, − 7), (10, − 4). **30.** (3, 8), (5, 4), (6, 2).

31. (− 5, − 5), (5, 2), (12, 7). **32.** (− 2, 12), (9, 6), (22, − 1).

Express the given statement by means of an algebraic equation. What is the locus of the point (x, y) in each case?

33. The point (x, y) is at the distance 3 from (6, − 2).
Ans. $x^2 + y^2 - 12x + 4y + 31 = 0$.

34. The point (x, y) is at the distance 2 from (− 5, 7).
Ans. $x^2 + y^2 + 10x - 14y + 70 = 0$.

35. The point (x, y) is equidistant from (4, 2) and (− 3, 3).
Ans. $7x - y = 1$.

36. The point (x, y) is equidistant from (− 3, 5) and (− 4, − 2).
Ans. $x + 7y = 7$.

Draw the following figures, with the y-unit twice as large as the x-unit.

37. The square with vertices (0, 0), (1, 0), (1, 1), (0, 1).

38. The right triangle of Ex. 12.

39. The isosceles triangle of Ex. 13.

40. The square of Ex. 15.

6. Division of a line-segment. Given two points $P_1 : (x_1, y_1)$, $P_2 : (x_2, y_2)$, let it be required to find the point $P : (x, y)$ lying in the line joining P_1, P_2 and so placed that the segment P_1P is a given fraction of the entire segment P_1P_2: say

$$P_1P = k \cdot P_1P_2.$$

By similar triangles,

$$\frac{M_1M}{M_1M_2} = \frac{P_1P}{P_1P_2} = k,$$

or

$$M_1M = k \cdot M_1M_2.$$

But

$$M_1M_2 = x_2 - x_1,$$

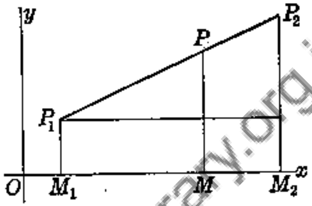

FIG. 5

so that

$$x = OM_1 + M_1M = x_1 + k(x_2 - x_1).$$

A similar formula for y is readily derived.

If P lies in the segment P_1P_2 produced, rather than in the interior of the segment, then obviously $k > 1$, but the result is the same. Hence:

If $P : (x, y)$ is a point in the line-segment P_1P_2 or in P_1P_2 produced, and if P is so placed that $P_1P = k \cdot P_1P_2$, then

$$(1) \qquad x = x_1 + k(x_2 - x_1), \qquad y = y_1 + k(y_2 - y_1).$$

Examples: (*a*) The segment from (1, 3) to (5, − 2) is trisected. Find the point of trisection nearer to (1, 3).

Here $k = \frac{1}{3}$, $x_2 - x_1 = 4$, $y_2 - y_1 = -5$, so that

$$x = 1 + \tfrac{1}{3} \cdot 4 = \tfrac{7}{3}, \qquad y = 3 + \tfrac{1}{3} \cdot (-5) = \tfrac{4}{3}.$$

Thus the required point is $(\frac{7}{3}, \frac{4}{3})$.

(*b*) A line is drawn from (5, − 4) to (7, − 9), and is then extended beyond the latter point so that its length is doubled. Find the terminal point.

Here $k = 2$, $x_2 - x_1 = 2$, $y_2 - y_1 = -5$, so that

$$x = 5 + 4 = 9, \qquad y = -4 - 10 = -14.$$

7. Midpoint of a line-segment. An important special case arises when P is the *midpoint* of the segment P_1P_2. Putting $k = \frac{1}{2}$ in (1), § 6, and simplifying, we find that:

The coördinates (x, y) *of the point midway between the points* (x_1, y_1), (x_2, y_2) *are*

$$(1) \qquad x = \tfrac{1}{2}(x_1 + x_2), \qquad y = \tfrac{1}{2}(y_1 + y_2).$$

EXERCISES

1. Trisect the line joining $(7, -2)$, $(3, 5)$. *Ans.* $(\frac{17}{3}, \frac{1}{3})$; $(\frac{13}{3}, \frac{8}{3})$.

2. The segment joining $(6, 14)$, $(2, -2)$ is to be divided into four equal parts. Find the points of division.

3. The segment joining $(5, 0)$, $(4, 3)$ is divided into two segments, one of which is three-fourths of the other. Find the point of division.

Ans. $(\frac{32}{7}, \frac{9}{7})$; $(\frac{31}{7}, \frac{12}{7})$.

4. The segment joining $(3, -2)$, $(10, 12)$ is divided into two segments, one of which is two-fifths of the other. Find the point of division.

5. The segment from $(-1, 4)$ to $(6, 2)$ is trebled. Find the endpoint.

6. The segment joining $(4, 3)$, $(6, -1)$ is extended each way a distance equal to one-half its own length. Find the endpoints.

7. Prove in two ways that the lines joining $(10, 4)$, $(3, -5)$, $(1, 1)$ form a right triangle. Find the area of the circumscribed circle.

8. Prove in two ways that the lines joining $(2, 4)$, $(-2, 3)$, $(8, -20)$ form a right triangle. Find the area of the circumscribed circle.

9. Prove in two ways that the lines joining $(\frac{3}{2}, 1)$, $(2, 5)$, $(3, -2)$, $(\frac{5}{2}, -6)$ form a parallelogram.

10. Prove in two ways that the lines joining $(1, 2)$, $(-5, 1)$, $(-6, -\frac{1}{2})$, $(0, \frac{1}{2})$ form a parallelogram.

11. Three consecutive vertices of a parallelogram are $(4, 2)$, $(5, 3)$, $(6, -4)$. Find the fourth vertex.

12. If $P : (x, y)$ divides the segment from $P_1 : (x_1, y_1)$ to $P_2 : (x_2, y_2)$ internally or externally in the ratio $r_1 : r_2$, i.e. if $\frac{P_1P}{PP_2} = \frac{r_1}{r_2}$, show that

$$x = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \qquad y = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}.$$

(Note that for external division, either r_1 or r_2 is negative.)

13. Solve the examples of § 6 by the formulas of Ex. 12.

14. Solve Exs. 1–6 by the formulas of Ex. 12.

8. Inclination; slope. The *angle of inclination*, also called simply the *inclination*, of a straight line is the (positive or negative) *acute angle* α *between the line and the positive x-axis*, this angle being measured *from Ox to* the line.

The *slope* of a line is the *tangent of the angle of inclination*. Slope is usually denoted by m:

(1) $$m = \tan \alpha.$$

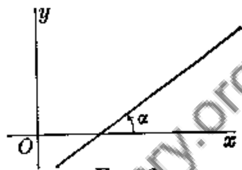

FIG. 6

With the axes in the usual position, a line sloping *upward to the right* has *positive slope*, since the tangent of a positive acute angle is positive; a line sloping *downward to the right* has *negative slope*.

For the line through $P_1 : (x_1, y_1)$, $P_2 : (x_2, y_2)$, we have

$$m = \tan \alpha = \frac{QP_2}{P_1Q}.$$

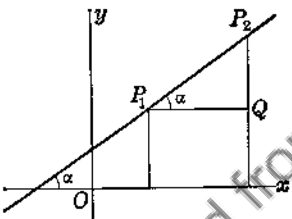

FIG. 7

Thus it follows from § 5 that:

The slope of the line joining the points $P_1 : (x_1, y_1)$, $P_2 : (x_2, y_2)$ *is*

(2) $$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

In ordinary language, the "slope" of a line means the ratio of "rise" to "run" — i.e. the *ratio of the vertical distance to the horizontal distance* covered in traversing any segment of the line. Thus a road with 10% "slope," or "grade," rises 10 ft. for every 100 ft. horizontally. This is exactly equivalent to the definition given above.

It appears from (2) that *lines parallel to the x-axis* (including the x-axis itself) *have the slope* 0. For in that case $y_2 = y_1$.

It should be noted that the idea of slope is meaningless in the case of a line parallel to the y-axis (including the

y-axis itself), since $\tan\alpha$ increases indefinitely as α approaches $90°$ — or, as we say, $\tan 90° = \infty$. For this reason, *in all discussions involving slopes, lines parallel to Oy are excluded.*

9. Parallel and perpendicular lines. If two lines are parallel, they have the same slope; and conversely.

Given two perpendicular lines λ_1, λ_2, with slopes

$$m_1 = \tan\alpha_1, \qquad m_2 = \tan\alpha_2,$$

let λ_2 denote the one with positive slope. Remembering that α_1 is negative, we see that $\alpha_2 = \alpha_1 + 90°$, so that

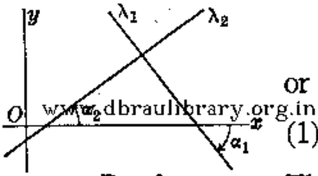

Fig. 8

$$\tan\alpha_2 = -\cot\alpha_1 = -\frac{1}{\tan\alpha_1},$$

or

$$(1) \qquad m_2 = -\frac{1}{m_1}.$$

Thus we have the

Theorem: *If two lines are perpendicular, the slope of one is the negative reciprocal of the slope of the other; or in other words, the product of their slopes is* -1.

Here also the converse is true; proof of the converse will be left to the student.

Example: Prove that the points $P_1 : (-1, 3)$, $P_2 : (0, 5)$, $P_3 : (3, 1)$ are the vertices of a right triangle.

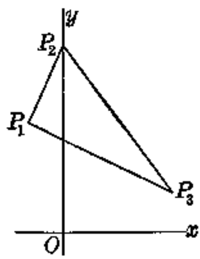

Fig. 9

From the figure we see that if there is a right angle it must be at P_1. The slopes of P_1P_2, P_1P_3 are respectively

$$m_1 = \frac{5-3}{0+1} = 2, \qquad m_2 = \frac{1-3}{3+1} = -\frac{1}{2},$$

whence it follows from the theorem (converse) that P_1P_2 and P_1P_3 are perpendicular.

10. Angle between two lines. By the *angle from* a line λ_1 *to* another line λ_2 we shall understand the *acute angle* through which λ_1 must be rotated to come to coincidence with λ_2. This angle is considered *positive if measured counter-clockwise, negative if measured clockwise.* In case we are concerned only with the magnitude of the angle without regard to sign, we may speak merely of the angle *between* the lines.

Let the lines λ_1, λ_2 make the (positive or negative) acute angles α_1, α_2 with Ox, and denote by ϕ the angle from λ_1 to λ_2. Then we have

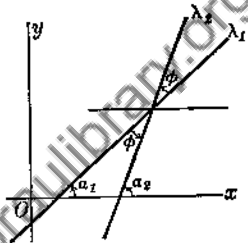

FIG. 10

$$\alpha_2 = \alpha_1 + \phi,$$

whence

$$\phi = \alpha_2 - \alpha_1,$$

and

$$\tan\phi = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2}.$$

But the slopes of the lines are

$$\tan\alpha_1 = m_1, \qquad \tan\alpha_2 = m_2,$$

so that *the angle between* two lines of slopes m_1, m_2 is given by the formula*

$$\textbf{(1)} \qquad \tan\phi = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

This result will be more easily remembered if we realize that it is not, properly speaking, a new formula at all, but merely a restatement of the familiar formula of trigonometry for the tangent of the difference of two angles.

Formula (1) fails if one line, say λ_2, is parallel to Oy, but in that case $\phi = 90° - \alpha_1$, so that $\tan\phi = \cot\alpha_1$.

* More precisely, the angle from the first line to the second.

EXERCISES

Find the slope of the line joining the given points. Draw the figure.

1. $(3, -4), (-1, 2)$. **2.** $(5, 3), (3, -2)$.

3. $(-6, 2), (2, -5)$. **4.** $(-7, -5), (-3, 0)$.

Test the following statements (Exs. 5–10) by methods based on §§ 8–9.

5. The lines joining $(1, 3)$, $(6, 5)$, $(5, -7)$ form a right triangle.

6. The lines joining $(2, 4)$, $(3, 8)$, $(5, 1)$, $(4, -3)$ form a parallelogram.

7. The lines joining $(-5, 3)$, $(7, -12)$, $(12, 24)$, $(0, 39)$ form a parallelogram.

8. The perpendicular bisector of the line joining $(-3, 1)$, $(13, 3)$ passes through $(7, -14)$.

9. The points $(1, -9)$, $(7, 6)$, $(3, -4)$ lie in a straight line.

10. The points $(0, 2)$, $(4, 1)$, $(16, -2)$ lie in a straight line.

11. Solve Ex. 31, p. 6, by a new method.

12. Solve Ex. 32, p. 6, by a new method.

13. Prove that the quadrilateral with vertices $(0, 1)$, $(4, 2)$, $(3, 6)$, $(-5, 4)$ has two right angles. Find the area. *Ans.* $A = \frac{51}{2}$.

14. Prove that the quadrilateral with vertices $(10, 10)$, $(-14, -2)$, $(-10, -10)$, $(4, -24)$ can be divided into two right triangles. Find the area.

15. Find the interior angles of the triangle of Ex. 5.

16. In Ex. 6, find the angle between two adjacent sides.

17. For the triangle of Ex. 31, p. 6, find the interior angles.

Ans. $\tan\phi_1 = \frac{1}{179}$, $\tan\phi_2 = \frac{1}{254}$, $\tan\phi_3 = -\frac{1}{105}$.

18. For the triangle of Ex. 32, p. 6, find the interior angles.

19. Using the formula $A = \frac{1}{2}bc\sin\alpha$, where α is the angle between the sides b, c, find the area of the triangle of Ex. 31, p. 6. (See Ex. 17.)

Ans. $\frac{1}{2}$.

20. Using the formula of Ex. 19, find the area of the triangle of Ex. 32, p. 6. (See Ex. 18.) *Ans.* $\frac{1}{2}$.

21. Prove the converse of the theorem of § 9.

11. Application of analysis to elementary geometry. The representation of points by pairs of numbers, which is effected by means of a Cartesian coördinate system, establishes a connection between algebra and geometry which enables us to *express the geometric properties of a figure in algebraic language*, and thus to solve geometric problems by purely algebraic (analytic) methods. Many important theorems of elementary geometry, particularly those involving polygons, can be proved analytically by the theory of the present chapter.

Example: Prove that the diagonals of a parallelogram bisect each other.

Let us place the parallelogram in the position shown. The coördinates of O are $(0, 0)$, those of P_1 may be taken as $(x_1, 0)$, those of P_2 as (x_2, y_2), whence those of P_3 *must be* $(x_1 + x_2, y_2)$. Thus the midpoint of P_1P_2 is $\left(\frac{x_1 + x_2}{2}, \frac{y_2}{2}\right)$; likewise the midpoint of OP_3 is $\left(\frac{x_1 + x_2}{2}, \frac{y_2}{2}\right)$, which proves the theorem.

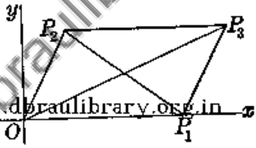

FIG. 11

It is obvious that *if the property to be proved is independent of the position of the figure, there is no loss of generality in assuming the figure in any convenient position* with reference to the coördinate axes. Thus the proof just given is entirely general. On the other hand, the figure itself must not be made special in any way: for instance, in the above example, by assigning numerical coördinates to the vertices, or by choosing as the given parallelogram a rhombus or a rectangle.

Further, this point must be carefully noted. While in elementary geometry the proof is obtained by studying the properties of the figure, in analysis the proof arises from algebraic work involving the coördinates: the figure is of no use at all beyond helping us to bear in mind the nature of the problem. Our notation must therefore be such that *the coördinates themselves express the data* of the problem. In the example, when coördinates have been assigned to three of the vertices, those of the fourth *are determined*, and must be correctly expressed in terms of those already assigned. If this is not done, the proof is impossible.

EXERCISES

Prove the following theorems.

1. The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

2. The diagonals of a rectangle are equal.

3. The distance between the midpoints of the non-parallel sides of a trapezoid is half the sum of the parallel sides.

4. An isosceles triangle has two equal medians.

5. A triangle having two equal medians is isosceles.

6. The medians of a triangle intersect in a trisection point of each.

7. If the diagonals of a rectangle are perpendicular, the rectangle is a square.

8. If a convex quadrilateral has two opposite sides equal and parallel, it is a parallelogram.

9. The lines joining the midpoints of the sides of a triangle divide it into four triangles of equal area.

10. A quadrilateral whose diagonals bisect each other is a parallelogram.

11. A quadrilateral whose diagonals bisect each other at right angles is a rhombus.

12. The diagonals of an isosceles trapezoid are equal.

13. A trapezoid whose diagonals are equal is isosceles.

14. The line segments joining the midpoints of opposite sides of a quadrilateral bisect each other.

15. The line segments joining the midpoints of adjacent sides of a quadrilateral form a parallelogram.

16. The sum of the squares on the sides of a parallelogram equals the sum of the squares on the diagonals.

CHAPTER II

THE LOCUS OF AN EQUATION

12. Constants; variables. In analytic geometry we deal with two kinds of quantities — "constants" and "variables."

A *constant* is a quantity whose value remains unchanged throughout any given problem. Examples are the coördinates of a fixed point, the radius of a given circle, the slope of a given line, etc. Coördinates of fixed points are usually denoted by the letters x, y with subscripts, as (x_1, y_1), (x_2, y_2), etc.; other letters, such as a, b, m, etc., are also used to denote constants.

A *variable* is a quantity that may take different values (usually an infinite number of them) in the same problem. The variables most frequently occurring are the coördinates (x, y) of a point moving along a definite path.

For example, the fact that a moving point (x, y) remains always at the constant distance a from the origin is expressed by the equation

(1) $$x^2 + y^2 = a^2.$$

This equation evidently remains true if the point (x, y) moves along the circle of radius a with center at O.

In elementary analytic geometry, all quantities occurring — both constants and variables — are restricted to *real values*. Thus, in (1), putting $x = \frac{1}{2}a$, we find $y = \pm \frac{1}{2}\sqrt{3}a$, which says that the points $(\frac{1}{2}a, \pm \frac{1}{2}\sqrt{3}a)$ are on the circle; putting $x = 2a$, we find $y = \pm\sqrt{3}ai$, which merely verifies the obvious fact that there is on the circle no point with abscissa $2a$.

13. The locus of an equation. Let two variables x and y be connected by an equation — for instance,

$$y = \tfrac{1}{2}x + 2, \qquad y^3 = 2x, \text{ etc.}$$

If any value be assigned to either variable, one or more values of the other are determined, in general, by the equation; thus there exist infinitely many pairs of values of x and y that satisfy the equation. Each pair of numbers may be represented geometrically by a point. The points so determined are not scattered at random throughout the plane, but form in the aggregate a definite *curve*. This curve is called the *locus* of the equation:

The locus of an equation is a curve containing those points, and only those points, whose coördinates satisfy the equation.

The curve corresponding to a given equation is said to *represent the equation geometrically*, while the equation *represents the curve analytically*. The study of plane curves by means of the equations representing them forms the subject-matter of plane analytic geometry.

The elementary method of tracing a curve is to find a number of pairs of values of x and y satisfying the equation, plot the corresponding points, and draw a smooth curve through them: this curve is approximately the desired locus. The great objection to this method is that it sheds but little light on the general properties of the curve. Better methods will be developed as we proceed.

It should be remarked that in exceptional cases an equation may represent only a single point, or it may have no locus whatever. For example, the equation $x^2 + y^2 = 0$ is evidently satisfied only by the coördinates (0, 0); the equation $x^2 + y^2 = -1$ is satisfied by no real values of x and y.

To plot a curve by points, we proceed as in the examples below. Sometimes it is convenient to adopt different scales in the two directions (see § 4). The scale or scales adopted should always be noted on the drawing.

Examples: (*a*) Trace the curve $y = \frac{1}{2}x + 2$.

We assign values to x and compute the values of y:

x	0	1	2	-1	-2	-3	-4	-5
y	2	$\frac{5}{2}$	3	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$

We now plot these points, choosing say two spaces on the coördinate paper as the unit, and draw a smooth curve through them. It should be noted that the curve picks up the points *according to the algebraic order of the values of x* — not as they are arranged in the above table. The "curve" in this case appears to be a straight line: it will be proved in § 26, and assumed meanwhile, that *the locus of every equation of the first degree is a straight line.*

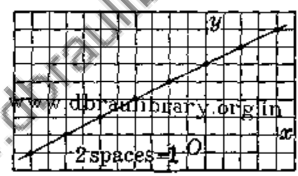

FIG. 12

(*b*) Plot the curve

$$y^3 = 2x.$$

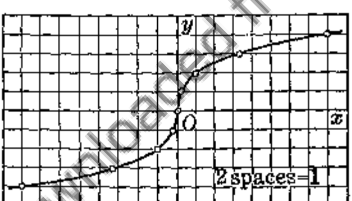

FIG. 13

Here, to avoid the extraction of cube roots, let us write the equation in the form

$$x = \frac{1}{2}y^3$$

and assign values to y. Since x becomes very large as y increases, we will assign moderately small values to y:

y	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2
x	0	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{27}{16}$	4	$-\frac{1}{16}$	$-\frac{1}{2}$	$-\frac{27}{16}$	-4

14. Intercepts on the axes. To find the points where the curve crosses Ox we must evidently *put* $y = 0$ *and solve for* x; to find the intersections with Oy we *put* $x = 0$ *and solve for* y. Of course in some cases the solution cannot be carried out on account of algebraic difficulties.

The directed distances cut off by a curve on the axes, measured from (0, 0), are called its x- and y-*intercepts*.

Example: The curve $3x^2 + 2y = 12$ has intercepts ± 2 on the x-axis and 6 on the y-axis: i.e. it passes through the points $(\pm 2, 0)$ and $(0, 6)$.

15. Symmetry. Two points P_1, P_2 are said to *be placed symmetrically*, or to *be symmetric*, with respect to a line, if that line is the perpendicular bisector of the segment P_1P_2; the line is then called an *axis of symmetry* (the line λ in Fig. 14). Each of the points P_1, P_2 is said to be the *image*, or *reflection*, of the other in the line λ. A curve or other plane figure is *symmetric with respect to an axis* * if, corresponding to every point P_1 of the figure, the image-point P_2 in that axis also belongs to the figure. This means that the figure is *unchanged by reflection* in the axis — i.e. that the part of the figure on one side of the axis is the image of the part on the other side. Figure 14 shows a curve symmetric with respect to the indicated line.

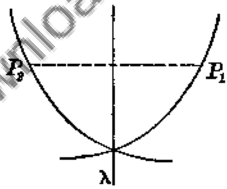

Fig. 14

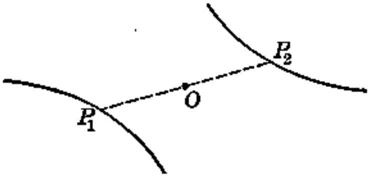

Fig. 15

* Obviously, this use of the word "axis" has nothing to do with "axis" of coördinates. There are two axes of coördinates, whereas a curve may have no axis of symmetry, or one, or more; further, if there is an axis of symmetry, this line need not be chosen as an axis of coördinates.

Two points P_1, P_2 are said to be placed *symmetrically with respect to a point O* if O is the midpoint of P_1P_2. A plane figure is symmetric with respect to a point O if, corresponding to every point P_1 of the figure, there is a point P_2, also belonging to the figure, such that O is the midpoint of P_1P_2. The point O is called the *center of symmetry*, or simply the *center*. In Fig. 15, the curve has the point O as center of symmetry.

16. Tests for symmetry. From the definitions of § 15 we deduce analytic tests for symmetry of a curve with respect to the coördinate axes and the origin.

THEOREM I: *A curve is symmetric with respect to the x-axis if its equation is unchanged * when y is replaced by − y; and conversely. A curve is symmetric with respect to the y-axis if its equation is unchanged when x is replaced by − x; and conversely.*

THEOREM II: *A curve is symmetric with respect to the origin if its equation is unchanged when x is replaced by − x and y by − y simultaneously; and conversely.*

Proof of I: By hypothesis, if any pair of coördinates (x, y) satisfy the equation, the coördinates $(x, -y)$ of the image-point with respect to the x-axis also satisfy the equation. The proof of the converse is left to the student.

The proof of II is also left as an exercise.

Example: Trace the curve

$$x^2 - y^2 = 4.$$

When $y = 0$, $x = \pm 2$; when $x = 0$, y is imaginary, so that the curve does not intersect Oy. The curve is symmetric to both axes.

* More precisely, if the new form of the equation is *equivalent* to the original form — i.e. if each form is satisfied by all the values of the variables that satisfy the other.

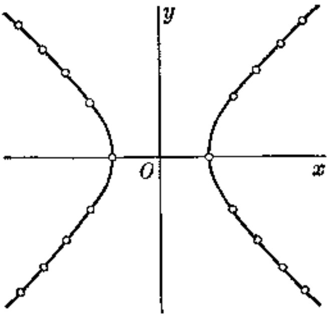

FIG. 16

Writing the equation in the form

$$y = \pm\sqrt{x^2 - 4},$$

we see that y is imaginary if x is numerically less than 2. Values of x greater than 2 give the following points:

x	3	4	5	6
y	$\pm\sqrt{5}$	$\pm 2\sqrt{3}$	$\pm\sqrt{21}$	$\pm 4\sqrt{2}$

On account of the symmetry with respect to Oy, the portion of the curve corresponding to negative values of x can be obtained by reflection in Oy.

EXERCISES

Draw the following straight lines. Plot three points in each case — the third as a check.

1. $2x - y = 6$.
2. $x + 3y = 3$.
3. $3x - 5y + 6 = 0$.
4. $5x - 4y + 10 = 0$.
5. $6x + 5y + 1 = 0$.
6. $x + 2y + 30 = 0$.
7. $3x - 5y = 0$.
8. $2x + 3y = 0$.
9. $x + 2 = 0$.
10. $y = 5$.

Find the intercepts on the axes, test for symmetry, plot a number of points, and trace the curve, choosing a suitable scale in each case.

11. $y = 2x^2 + 3$.
12. $y = 1 - 4x^2$.
13. $y = x^2 - 3x$.
14. $y = 3 + 2x - x^2$.
15. $x + y + y^2 = 0$.
16. $x - 2y^2 - 4y = 0$.
17. $x^2 + 2y^2 = 2$.
18. $x^2 + 3y^2 = 9$.
19. $9x^2 + 4y^2 = 1$.
20. $4x^2 + y^2 = 400$.
21. $4x^2 - y^2 = 4$.
22. $x^2 - 2y^2 = 2$.
23. $x^2 - y^2 + 1 = 0$.
24. $x^2 - 4y^2 + 16 = 0$.
25. $y = x^3 - 5x^2 + 6x$.
26. $y = x^3 + 7x^2 + 6x$.
27. $y = x^3 - 2x^2 - 9x + 18$.
28. $y = -x^3 + 3x^2 - 4$.
29. $x = (y^2 - 1)^2$.
30. $x = y^2(4 - y^2)$.

31. Without formal proofs, state how many axes of symmetry are possessed by (*a*) a circle; (*b*) a circular arc; (*c*) a straight line; (*d*) a line segment; (*e*) a square. *Ans.* (*d*) Two.

32. Show that two circles taken together always have one axis of symmetry. When are there two such axes? When more than two?

33. When do three circles have one or more axes of symmetry?

34. When do two circles have a center of symmetry?

35. Prove analytically that if a curve is symmetric with respect to Ox and Oy, it is symmetric with respect to the origin. Show by an example that the converse is not true.

36. Prove the theorem of Ex. 35 geometrically.

37. Prove that if a curve is symmetric to one coördinate axis and the origin, it is symmetric to the other coördinate axis also.

17. Consequences of the definition of locus. From the definition of locus of an equation (§ 13), it follows that a point lies on a curve *if and only if its coördinates satisfy the equation of the curve.* This suggests

RULE I: *To find out whether a point lies on a given curve, substitute its coördinates for x and y in the equation of the curve.*

Example: (*a*) The point (2, 12) lies on the curve

$$y = 3x^2$$

because

$$12 = 3 \cdot 4;$$

the point (– 1, – 3) does not lie on the curve because

$$-3 \neq 3 \cdot 1.$$

RULE II: *To express analytically the condition that a point shall lie on a curve, write the equation of the curve with the coördinates of the point substituted for x and y.*

Examples: (*b*) The point (x_1, y_1) lies on the curve

$$y^2 = 4x$$

if and only if

$$y_1^2 = 4x_1.$$

(c) Determine the constant a so that the point $(-2, -3)$ shall lie on the curve

$$y = ax^2.$$

Substituting the coördinates $(-2, -3)$, we get

$$-3 = 4a, \quad \text{or} \quad a = -\tfrac{3}{4},$$

and the equation of the curve is

$$y = -\tfrac{3}{4}x^2.$$

RULE III: *To find the ordinate of a point on a curve when the abscissa is given, substitute the given abscissa for x in the equation of the curve and solve for y.* Similarly we may find the abscissa when the ordinate is given.

18. Factorable equations. In algebra, when the product of two or more factors is equal to 0, the equation thus formed is satisfied whenever any one of the factors is 0. For instance, the equation

(1) $$3(x + 2y)(x^2 - y^2) = 0$$

is satisfied if and only if

(2) $$x + 2y = 0,$$

(3) $$x + y = 0,$$

or

(4) $$x - y = 0.$$

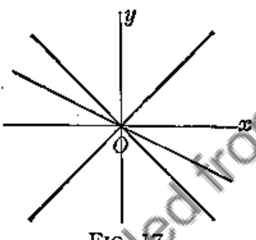

FIG. 17

Hence, the locus of an equation whose right member is 0 and whose left member can be factored consists of all points whose coördinates when substituted in the equation *cause any one of the factors to vanish*. That is, the curve represented by such an equation breaks up into *several distinct curves*, viz. the loci of the equations formed by equating the several factors separately to 0. Thus equation (1) above represents (Fig. 17) the three straight lines (2), (3), (4).

19. Points of intersection of two curves. The points of intersection of two curves are points whose coördinates *satisfy both equations.* There are no other points having this property. Hence *the points of intersection of two curves are found by solving the equations of the curves as simultaneous equations.*

It may happen that all the values of x and y, found by solving two simultaneous equations, are imaginary; or the equations may be "incompatible" — i.e. not satisfied by any pairs of values either real or imaginary. In either case the result means that the curves do not intersect.

After solving two simultaneous equations the results should always be tested by substituting the values of x and y in both equations and noting whether the equations hold.

Example: Find the points of intersection of the line

(1) $$2x + y = 10$$

and the circle

(2) $$x^2 + y^2 = 25.$$

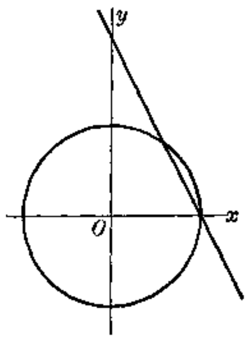

FIG. 18

Substituting the value of y from (1) in (2), we get

$$x^2 + (10 - 2x)^2 = 25,$$

or

$$5x^2 - 40x + 75 = 0,$$
$$x^2 - 8x + 15 = 0,$$

whence

$$x = 3 \text{ or } 5.$$

By (1),

$$y = 4 \text{ or } 0,$$

and the points are (3, 4), (5, 0).

Check:

$$6 + 4 = 10, \qquad 9 + 16 = 25;$$
$$10 + 0 = 10, \qquad 25 + 0 = 25.$$

The *degree* of a term involving x and y is the sum of the exponents of x and y in the term. Thus the terms x^2, $3xy$, $2y^2$ are of the second degree in x and y; $3x^4$, $2xy^3$ are of the fourth degree.

The *degree* of an algebraic equation in x and y is that of the term of highest degree when the equation is rationalized and cleared of fractions.* A curve whose equation is of the nth degree is called a *curve of the nth degree.*

When two equations in x and y are solved as simultaneous, the number of solutions (i.e. the number of pairs of values of x and y satisfying both equations) is not greater than the product of the degrees of the equations. Hence *the number of points common to two curves is not greater than the product of the degrees of their equations.*

EXERCISES

Determine whether the given points lie on the given curve.

1. Curve $4x + y = 6$; points $(1, 2)$, $(2, -1)$, $(4, -10)$, $(0, -6)$.
2. Curve $2x - 3y + 4 = 0$; points $(1, 3)$, $(-2, 0)$, $(7, 6)$, $(10, 8)$.
3. Curve $y = x^3 - 4x^2 + 2x + 5$; points $(-1, 6)$, $(2, 1)$, $(3, 2)$.
4. Curve $x^2 - xy + 2y^2 + 6y = 4$; points $(2, -2)$, $(-2, 0)$, $(4, 2)$.
5. Curve $y^2 = 4ax$; points $(2a, a)$, $(-a, -2a)$, $(4a, -4a)$, $(\frac{1}{4}a, a)$.
6. Curve $x^2 + y^2 = 5$; points $(-2, -1)$, $(\sqrt{2}, \sqrt{3})$, $(1.41, 1.73)$.
7. Determine k so that the straight line $x + 2y = k$ shall pass (*a*) through $(4, 1)$; (*b*) through $(-6, -4)$; (*c*) through $(0, 0)$.
8. For what value of a does the curve $y = ax^3$ pass through $(-1, -3)$? Through $(4, 2)$? Through $(0, 0)$? Through $(0, 2)$? Through $(2, 0)$?
9. Determine m so that the line $y = mx + 2$ shall pass (*a*) through $(1, 3)$; (*b*) through $(2, -2)$; (*c*) through $(0, 2)$; (*d*) through $(0, 0)$.
10. What is the condition that the curve $y = ax^2 + bx + c$ shall pass through $(0, 0)$? Through $(2, 1)$? Through $(-1, c)$?
11. Determine A and B so that the line $Ax + By = 1$ shall pass through the points $(3, -2)$, $(4, 2)$.

*As regards the variables only: irrational and fractional constants may be present.

12. Determine A and B so that the line $Ax + By = 1$ shall pass through the points $(7, 2)$, $(1, -4)$.

13. Determine a and b so that the curve $x^2 + y^2 + ax + by = 0$ shall pass through the points $(1, 2)$, $(-3, 3)$. *Ans.* $a = \frac{7}{3}$, $b = -\frac{11}{3}$.

14. On the curve $y^2 = x^3$, find (*a*) the point whose ordinate is -1; (*b*) the points whose abscissa is 4.

15. On the curve $y^2 - x - 3y + 2 = 0$, find (*a*) the point whose ordinate is 3; (*b*) the points whose abscissa is 0; (*c*) the points whose abscissa is 3; (*d*) the points whose abscissa is -4.

16. On the curve $y^2 = x^3 - 2x^2 - 5x + 7$, find the points (*a*) whose abscissa is -1; (*b*) whose ordinate is -1; (*c*) whose abscissa is 2.

Trace the following curves, after factoring the equations (§ 18).

17. $x^2 + 3xy = 0$.

18. $x^3y = 4xy^3$.

19. $x^2 - 4xy + 4y^2 = 0$.

20. $x^2 - 4xy + 4y^2 = 4$.

21. $xy^2 = x^2$.

22. $y^2 = x^4$.

Find the points of intersection of the given curves; check by substitution in the given equations; plot the two curves on the same axes.

23. $4x + 4y = 5$, $x - 2y + 1 = 0$.

24. $x^2 + y^2 = 5$, $3x - y = 5$.

25. $x^2 + y^2 = 10$, $3x + y = 10$.

26. $x^2 + y^2 = 5$, $x^2 = 2y + 5$.

27. $y = x^3 - 6x^2 + 4x$, $8x + y = 8$. *Ans.* $(2, -8)$ three times.

28. $y = x^3 - 2x^2 - 5$, $x + y + 5 = 0$.

29. $y = x^3$, $x + y + 10 = 0$.

30. $y = 2x^2 - 3x + 1$, $y = x^3 - 4x^2 + 2x + 1$.

31. $4y = -4x^3 - 3x^2 + 14x + 8$, $4y = 4x^3 - 3x^2 - 18x + 8$.

32. $y = x^3$, $2x = 3y - y^2$. *Ans.* $(0, 0)$, $(1, 1)$, $(1, 1)$, $(-2, 4)$.

33. $y = x^2$, $y^2 + 6x - 7y = 0$. *Ans.* $(0, 0)$, $(1, 1)$, $(2, 4)$, $(-3, 9)$.

CHAPTER III

THE EQUATION OF A LOCUS

20. Path of a moving point. The work of Chapter I illustrated in a preliminary way the basic reason for the existence of analytic geometry — viz. that the introduction of a coördinate system makes it possible to translate geometric problems into algebraic language, and to solve them by purely algebraic methods.

In Chapter II we took a long step forward, when it appeared that a *plane curve* may be represented algebraically by an *equation* in x and y, whence the geometric properties of the curve may be discovered by merely interpreting geometrically the algebraic properties of the equation. In that chapter, however, the equation of the curve was always given in advance. Unfortunately, in practice the equation is very seldom given to begin with; it is usually necessary, as a first step, to *derive* the equation from a geometric definition of some kind. This problem will occupy us in the present chapter.

Very often the curve is defined as the *path*, or *locus, of a point which moves according to a given law.* In such a case, the statement of the law of motion usually suggests an equality between certain distances or other quantities involving the coördinates of the moving point. We always *denote the coördinates of the moving point by* (x, y), and try to obtain (by the formulas of analytic geometry) expressions for the distances or other quantities involved in the statement of the law of motion, after which it is usually a simple matter to write out the equation of the locus.

Examples: (*a*) A point moves so as to remain always equidistant from the points $P_1 : (3, 2)$ and $P_2 : (-1, 5)$. Find the equation of its locus.

According to the statement of the problem, the point $P : (x, y)$ moves so that $PP_1 = PP_2$. Hence, by § 5,

$$\sqrt{(x-3)^2 + (y-2)^2} = \sqrt{(x+1)^2 + (y-5)^2}.$$

This equation expresses the given law of motion and is therefore *the equation of the locus.** Squaring and expanding, we get

$$x^2 - 6x + 9 + y^2 - 4y + 4 = x^2 + 2x + 1 + y^2 - 10y + 25,$$

or

$$8x - 6y + 13 = 0,$$

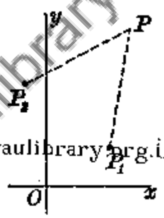

FIG. 19

which shows that the locus is a straight line. It is of course the perpendicular bisector of the line joining the fixed points.

(*b*) A point moves so as to remain equidistant from the line $y = -1$ and the point $P_1 : (0, 1)$. Find the equation of its locus.

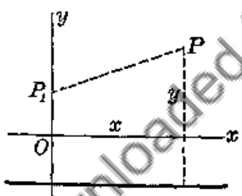

FIG. 20

The distance of any point $P : (x, y)$ from the line $y = -1$ is $y + 1$, as is evident from the figure. Hence we have

$$y + 1 = \sqrt{x^2 + (y-1)^2},$$
$$y^2 + 2y + 1 = x^2 + y^2 - 2y + 1,$$
$$x^2 = 4y.$$

* To prove strictly that a curve and its equation as found in the present chapter correspond to each other in the manner required by the definition of "locus of an equation" (§ 13), it is clearly necessary to show (*a*) that the coördinates of every point of the locus satisfy the equation; (*b*) that every point whose coördinates satisfy the equation lies on the locus. The method outlined above establishes only the first of these statements. However, in most cases the second part of the proof can be carried out by merely reversing the steps already taken, and therefore will usually be omitted.

EXERCISES

In each of the following cases, find the equation of the locus.

1. A moving point is always equidistant from the points (3, 1), (− 2, 5). *Ans.* $8y = 10x + 19$.

2. A moving point is always equidistant from (− 2, 2), (5, − 3).

3. A point moves so that its distance from (4, 0) is always twice its distance from (1, 0). Draw the curve. *Ans.* $x^2 + y^2 = 4$.

4. A point moves so that its distance from (0, 4) is two-thirds of its distance from (0, 9). Draw the curve. *Ans.* $x^2 + y^2 = 36$.

5. A moving point is always at the distance 4 from (1, − 2).

6. A moving point is always equidistant from the line $y = 3$ and the point (0, − 3). *Ans.* $x^2 + 12y = 0$.

7. A moving point is always equidistant from the y-axis and the point (4, 0). *Ans.* $y^2 = 8x - 16$.

8. A moving point is always equidistant from the line $x + 3 = 0$ and the point (2, 1). *Ans.* $y^2 - 10x - 2y = 4$.

9. A point moves so that its distance from the point (0, 9) is three times its distance from the line $y = 1$. *Ans.* $8y^2 - x^2 = 72$.

10. A point moves so that its distance from the point (1, 0) is one-half of its distance from the line $x = 4$. *Ans.* $3x^2 + 4y^2 = 12$.

11. A point moves so that its distance from (0, 5) is two-thirds of its distance from the x-axis. *Ans.* $9x^2 + 5y^2 - 90y + 225 = 0$.

12. A point moves so that its distance from (3, 2) is twice its distance from the y-axis. *Ans.* $3x^2 - y^2 + 6x + 4y = 13$.

13. A point moves so that the sum of the squares of its distances from (1, 0), (− 1, 0) is 10. Draw the curve. *Ans.* $x^2 + y^2 = 4$.

14. A point moves so that the sum of the squares of its distances from (0, 3), (0, − 3) is 20. Draw the curve. *Ans.* $x^2 + y^2 = 1$.

15. A point moves so that the sum of its distances from (4, 0) and (− 4, 0) is 10. *Ans.* $9x^2 + 25y^2 = 225$.

16. A point moves so that the sum of its distances from (0, 1) and (0, − 1) is $2\sqrt{2}$. *Ans.* $2x^2 + y^2 = 2$.

17. A point moves so that the difference of its distances from (0, 0) and (0, 8) is 4.

18. Two vertices of a right triangle are (1, 1), (4, 3), with the right angle at (1, 1). Find the locus of the third vertex by using § 9.

19. Solve Ex. 18 by using § 5.

20. Solve Ex. 18 by using § 7.

21. Two vertices of a right triangle are (0, 6), (5, 2), with the right angle at (0, 6). Find the locus of the third vertex in three ways.

22. The hypotenuse of a right triangle joins the points (5, − 1), (3, 3). Find the locus of the third vertex in three ways. (§§ 5, 7, 9.)

21. Loci defined geometrically. Sometimes we have a curve actually drawn, and are required to find its equation from the known geometric properties of the figure, as in example (*a*) below; or a geometric construction may be given by means of which the points of the curve are determined, as in (*b*).

In such cases we *assume a point of the curve in a general position, and denote its coördinates by* (x, y). Then the problem is merely to express some characteristic property of the curve by means of an equation involving x, y, and the constants of the problem. (By "characteristic property" is meant, of course, one that holds for all the points of the curve, and for no other points.) No matter what property is used, the result must be the equation of the curve.

Examples: (*a*) Find the equation of the straight line whose intercepts on the axes are $OA = 3$ and $OB = 2$.

Assume a point $P:(x, y)$ in a general position on the line. It is clear that the triangles MAP and OAB are similar if and only if P is on the line. Hence if the fact that these triangles are similar be expressed by an equation involving x and y, that equation must represent the line. Now

$$\frac{MP}{OB} = \frac{MA}{OA};$$

that is,

$$\frac{y}{2} = \frac{3 - x}{3}.$$

Simplifying, we get the form

$$2x + 3y = 6.$$

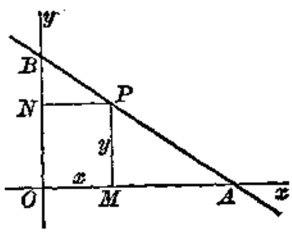

FIG. 21

(*b*) Find the locus of the centers of circles passing through the point $P_1:(0,\ 1)$ and tangent to the line $y = -1$.

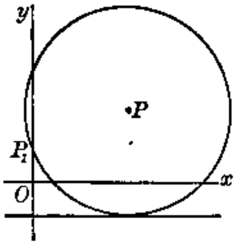

FIG. 22

Let us draw one* of the circles in question, and let its center P be the point (x, y). We note that P is equidistant from the line $y = -1$ and the point $(0, 1)$, so that the locus is the same as in example (*b*), § 20:

$$y + 1 = \sqrt{x^2 + (y - 1)^2}, \text{ etc.}$$

EXERCISES

1. Find the equation of the perpendicular bisector of the line joining $(4, 1)$, $(5, -3)$.

2. Find the equation of the perpendicular bisector of the line joining $(0, 5)$, $(4, -3)$. *Ans.* $2y = x$.

3. Find the locus of the centers of circles passing through the points $(1, 5)$, $(-3, -1)$. *Ans.* $2x + 3y = 4$.

4. Find the locus of the centers of circles passing through the points $(-1, -3)$, $(3, -1)$. *Ans.* $2x + y = 0$.

5. The base of an isosceles triangle is the line from $(7, 3)$ to $(-7, 1)$. Find the locus of the third vertex by two methods.

6. The base of an isosceles triangle joins the points $(0, 4)$, $(-6, 0)$. Find the locus of the third vertex by two methods.

7. Solve example (*a*) by making use of the fact that the triangles *MAP* and *NPB* are similar.

8. A circle is drawn having the points $(1, 1)$, $(5, -1)$ as ends of a diameter. Find its equation by three methods.

9. A circle is drawn having the points $(4, 1)$, $(2, 3)$ as ends of a diameter. Find its equation by three methods.

10. Find the locus of the centers of circles passing through $(2, 5)$ and tangent to the line $x + 2 = 0$.

11. Find the locus of the centers of circles passing through $(-3, -5)$ and tangent to the line $y + 3 = 0$.

* To draw more than one would merely obscure the figure, without helping us to find the equation of the locus.

12. One of the equal sides of an isosceles triangle is the line from (4, 2) to (3, 3). Find the locus of the third vertex.

Ans. $x^2 + y^2 - 6x - 6y + 16 = 0$; $x^2 + y^2 - 8x - 4y + 18 = 0$.

13. A variable circle is tangent externally to two fixed circles of radius 1 with centers at (5, 0) and (1, 7). Find the locus of the center of the variable circle. *Ans.* $8x - 14y + 25 = 0$.

14. A moving circle is tangent to the y-axis and to a circle of radius 1 with center at (2, 0). Find the locus of the center of the moving circle. Draw the curve. *Ans.* $y^2 - 6x + 3 = 0$; $y^2 - 2x + 3 = 0$.

15. A moving circle is tangent to the x-axis and to a circle of radius 2 with center at (0, 4). Find the locus of the center of the moving circle. Draw the curve.

16. Two circles are drawn: one of radius $\sqrt{5}$ with center at (1, 3), the other of radius $2\sqrt{5}$ with center at (0, 1). Find the equation of their common tangent. (§ 6.) *Ans.* $x + 2y = 12$.

17. A line segment of length $2a$ moves with its ends in the coördinate axes. Find the locus of its midpoint. *Ans.* $x^2 + y^2 = a^2$.

18. A line segment of length $3a$ moves with its ends in the coördinate axes. Find the locus of each of its points of trisection. Draw the curves.

Ans. $x^2 + 4y^2 = 4a^2$; $4x^2 + y^2 = 4a^2$.

19. Two circles of radius 10 are drawn with centers at (7, 1) and $(-7, 3)$. Find the equation of their common chord, and the length of the chord. *Ans.* $y = 7x + 2$; $10\sqrt{2}$.

20. Two circles of radius 7 are drawn with centers at (3, 2), $(-5, -4)$. Find the equation of their common chord, and the length of the chord.

CHAPTER IV

THE STRAIGHT LINE

22. Line parallel to a coördinate axis. If a straight line is parallel to the y-axis, its equation is

$$x = k,$$

where k is the (directed) distance of the line from the axis; and conversely. For all points on that line, and no other points, have the abscissa k.

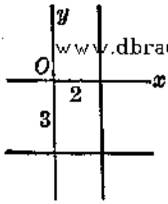

FIG. 23

Similarly, a line parallel to and at a directed distance k from the x-axis has the equation

$$y = k.$$

Figure 23 shows the lines $x = 2$, $y = -3$.

23. Line through a given point in a given direction: point-slope form. Let us try to find the equation of a straight line passing through a given point $P_1 : (x_1, y_1)$ in a given direction — i.e. having a given slope $m = \tan \alpha$.

Assuming a point $P : (x, y)$ in a general position on the line (cf. § 21), we note that, in the triangle P_1MP,

$$\tan \alpha = \frac{MP}{P_1M} = m;$$

that is,

$$\frac{y - y_1}{x - x_1} = m,$$

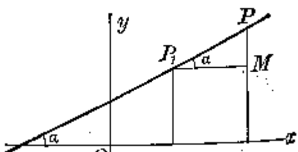

FIG. 24

or

(1) $$y - y_1 = m(x - x_1).$$

This is the *point-slope form* of the equation of the straight line. It is one of several so-called *standard forms* which will be developed in this chapter.

The point-slope form fails in case the line is parallel to the y-axis, since for such a line $m = \tan 90°$, which is nonexistent. (Note the remarks in the last paragraph of § 8.) But by § 22, the equation of a line through (x_1, y_1) parallel to Oy is simply

$$x = x_1.$$

Example: Find the equation of the line through $(3, -6)$ perpendicular to the line joining $(4, 1)$ and $(2, 5)$.

By § 8, the slope of the line joining $(4, 1)$ and $(2, 5)$ is -2, whence (§ 9) the slope of the required line is $\frac{1}{2}$, and by (1) its equation is *

$$y + 6 = \tfrac{1}{2}(x - 3),$$

or

$$x - 2y = 15.$$

When the slope and one point of a line are given, the line can be drawn, if desired, *without writing the equation* (see § 8). For example, to draw the line of slope $\frac{3}{2}$ through the point $(4, 2)$: starting at $(4, 2)$, we measure off 2 units to the right and then 3 upward (or 4 to the right and 6 upward, or 2 to the left and 3 downward, etc.); through the point thus reached and $(4, 2)$ we draw the line.

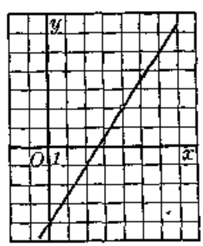

FIG. 25

* It is hardly necessary to remark that this example could be solved, and in fact similar problems have been solved, by the fundamental process of Chapter III (e.g. Exs. 18–20, p. 28). In establishing formula (1) we have merely generalized the process once for all, and in future will use the formula as a timesaver, just as in algebra we employ a formula for the roots of a quadratic equation. The same remark applies to many other formulas later.

24. Two-point form. To find the equation of the straight line through two given points $P_1 : (x_1, y_1)$ and $P_2 : (x_2, y_2)$, we note that its slope, by § 8, is

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \qquad (x_2 \neq x_1)$$

Substituting this value of m in (1), § 23, we obtain the *two-point form* of the equation of the straight line:

$$(1) \qquad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Alternate method: The equation of the line through any two points (x_1, y_1), (x_2, y_2) may be written in the form

$$(2) \qquad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

For, first, upon expanding the determinant by minors of the first row, we see that the equation is of the first degree,* so that its locus is a straight line. Second, upon substituting the coördinates of either of the given points for x and y (Rule I, § 17), we obtain a determinant with two identical rows, which therefore vanishes.

It may be noted that formula (2) holds in all cases — even when the line is parallel to Oy.

Example: Find the equation of the line through the points $(-6, -2)$, $(3, -5)$.

Substituting in (1), we find

$$(y + 5) = -\tfrac{3}{9}(x - 3),$$

which reduces to

$$x + 3y + 12 = 0.$$

Check: $3 - 15 + 12 = 0, \qquad -6 - 6 + 12 = 0.$

*Of course this result appears even more directly from the fact that in the expansion of a determinant, each term contains one and only one element from each row and each column.

EXERCISES

Draw the following lines.

1. $x = 0$. **2.** $y = 0$. **3.** $y + 4 = 0$.

4. $y = 5$. **5.** $x - 6 = 0$. **6.** $x + 7 = 0$.

Draw the following lines; then write the equations by (1), § 23.

7. Of slope $\frac{2}{3}$ through (4, 1).

8. Of slope $-\frac{5}{7}$ through $(-1, -2)$.

9. Through (1, 5) (*a*) parallel, (*b*) perpendicular to the line through $(4, -1)$, (3, 2).

10. Through $(3, -4)$ (*a*) parallel, (*b*) perpendicular to the line through $(0, -5)$, $(4, -3)$.

11. Through (5, 0) (*a*) parallel, (*b*) perpendicular to the line through $(-2, -4)$, $(-3, -2)$.

12. Through $(-3, 6)$ (*a*) parallel, (*b*) perpendicular to the line through $(-3, -4)$, $(5, -1)$.

Write the equation of the line by two methods. Check the answer.

13. Through (1, 0), (4, 2).

14. Through (5, 3), $(-1, 5)$.

15. Through $(-1, -2)$, $(-3, 5)$.

16. Through $(2, -3)$, $(-3, -4)$.

17. Find the equations of the medians of the triangle whose vertices are (5, 3), $(7, -1)$, $(3, -3)$. Find their point of intersection by § 6.

18. In the triangle with vertices (2, 0), (3, 2), $(4, -3)$, find the equations of the altitudes, and their point of intersection. *Ans.* $(-\frac{6}{7}, -\frac{4}{7})$.

25. Slope-intercept form. Given a line of slope m whose y-intercept is b, let us assume a point $P : (x, y)$ on the line. Then

$$m = \frac{MP}{QM} = \frac{y - b}{x},$$

whence

(1) $$y = mx + b.$$

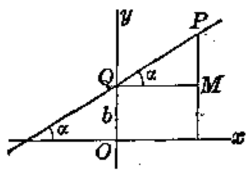

FIG. 26

Formula (1) is a special case of the point-slope form (§ 23) — viz. the case $x_1 = 0$, $y_1 = b$. It is known as the *slope-intercept form.*

26. The general equation of first degree. If an equation is of the first degree in x and y, it can contain at most a term in x, a term in y, and a constant term: i.e. it can be written in the form

(1) $$Ax + By + C = 0.$$

If $B = 0$, the equation evidently represents a line parallel to Oy. If $B \neq 0$, the equation can be solved for y: i.e. it can be written in the form

(2) $$y = mx + b.$$

Now for any value of m and any value of b there exists a line with slope m and y-intercept b; thus it follows from § 25 that equation (2) represents a line. Hence we have proved the

THEOREM: *The locus of every equation of the first degree is a straight line.*

The converse of this theorem is also true:*

The equation of a straight line is always of the first degree.

For, if the line is parallel to the y-axis, its equation has the form

$$x = k,$$

which is of the first degree. If it intersects the y-axis, it must have a certain slope and a certain y-intercept (either or both may of course be 0), and by § 25 its equation may be written in the form

$$y = mx + b,$$

which is also of the first degree.

To reduce the equation of any line (not parallel to Oy) to the slope-intercept form, *solve the equation for y.* When

*A highly critical reader might take exception to the converse theorem. Take for example the equations

$$x - y = 0, \qquad x^2 - 2xy + y^2 = 0, \qquad (x - y)(x^2 + y^2 + 1) = 0.$$

These equations all have the same locus, a straight line, yet only one is of the first degree. However, it is natural to take the simplest — i.e. the one of first degree — as "the" equation of the line, and we will agree to do this. A similar remark applies to various theorems later.

this has been done, *the coefficient of x is the slope and the constant term is the y-intercept.*

Example: The equation

$$3x + 4y + 6 = 0$$

becomes, in the slope form,

$$y = -\tfrac{3}{4}x - \tfrac{3}{2},$$

whence the slope is $-\frac{3}{4}$ and the y-intercept is $-\frac{3}{2}$.

27. Parallel and perpendicular lines. By reduction to the slope form, it is easily seen that the lines

$$Ax + By + C = 0,$$
$$Ax + By + K = 0$$

are parallel, while the lines

$$Ax + By + C = 0,$$
$$Bx - Ay + K = 0$$

are perpendicular. Hence, if a line is to be parallel to a given line, the coefficients of x and y in the required equation may be taken *the same as those in the given equation;* if a line is to be perpendicular to a given line, the coefficients of x and y in the required equation may be found by *interchanging the coefficients of x and y and changing the sign of one of them.* In each case, of course, the constant term must be determined by an additional condition.

Example: Write the equation of a line through the point $(3, -1)$ perpendicular to the line $3x + 2y = 6$.

The left member of the required equation will be $2x - 3y$; if the new equation is to be satisfied by the coördinates $(3, -1)$, the right member must be what the left member becomes when those coördinates are substituted. Hence the required equation is

$$2x - 3y = 9.$$

EXERCISES

Reduce the following equations to the slope form, and draw the lines.

1. $3x - 2y + 4 = 0$.
2. $x + 3y - 5 = 0$.
3. $4x + 5y + 1 = 0$.
4. $2x - 5y + 3 = 0$.
5. $7x - 4y = 0$.
6. $2x + 3y = 0$.

Write, at sight, the equations of the following lines. (§ 27.)

7. Through $(-1, 5)$ (*a*) parallel, (*b*) perpendicular to the line $2x - 3y = 4$.

8. Through $(4, -1)$ (*a*) parallel, (*b*) perpendicular to the line $3x + 4y + 1 = 0$.

9. Having the x-intercept 3 and (*a*) parallel, (*b*) perpendicular to the line $2x - 5y = 5$.

10. Having the x-intercept -1 and (*a*) parallel, (*b*) perpendicular to the line $x - 6y = 1$.

11. Through $(-1, 2)$ (*a*) parallel, (*b*) perpendicular to the line $2y = 5$.

12. Show that the lines $x + 2y + 1 = 0$, $6x - 3y = 5$, $y = 2x - 1$, and $4x + 8y + 7 = 0$ form a rectangle.

13. Find the locus of centers of circles tangent to the line $2x + 3y = 5$ at $(4, -1)$.

14. Find the locus of centers of circles tangent to the line $3x + y = 9$ at $(1, 6)$.

15. Show that a circle can be drawn tangent to the lines $y = x + 4$ and $7x + y = 8$ at $(3, 7)$ and $(1, 1)$ respectively.

16. Can a circle be drawn tangent to the lines $x + y = 2$ and $7x + 20y = 9$ at $(-1, 3)$ and $(7, -2)$ respectively?

Find the angle between the given lines. (§ 10.)

17. $3x + 4y = 0$, $5x - 2y = 3$. *Ans.* $\tan \phi = \frac{26}{7}$.

18. $x - 3y = 2$, $3x + 2y = 4$. *Ans.* $\tan \phi = \frac{11}{3}$.

19. $x + 4y = 3$, $5y = 3x + 2$.

20. $4x - 3y = 6$, $x - 7y = 2$.

21. $3x + 1 = 0$, $2x + 3y = 6$. *Ans.* $\tan \phi = \frac{3}{2}$.

22. $4x = 3$, $5x - 2y = 2$. *Ans.* $\tan \phi = \frac{2}{5}$.

28. Linear functions. If two variables y and x are so related that, *when the value of x is given, the value of y is determined*, then y is said to be a *function of x*.

Countless examples of functional relationship are already familiar to the student. The volume of a sphere is a function of — in other words, is determined by — the radius; the strength of a piece of copper wire depends upon its size; the value of a diamond of given quality depends upon its weight.

When x and y are connected by an equation of the *first degree*, we say that y is a *linear function* of x. Every linear function may be written in the form

$$y = mx + b.$$

For the present, we shall be concerned only with linear functions; examples of other types will occur later.

The variable x is called the *independent variable*. It is evidently possible to reverse the roles of the two variables: by merely solving for x, we may express x as a function of y.

Examples: (*a*) A car drives from A to B, 200 miles distant, at 40 mi. per hr. The distance from A (in miles) after time t (in hours) is

$$s = 40t.$$

(*b*) In (*a*), the distance from B is

$$s = 200 - 40t.$$

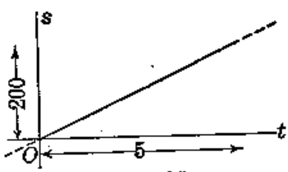

FIG. 27

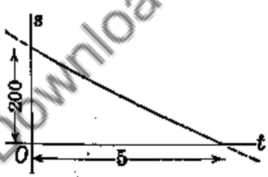

FIG. 28

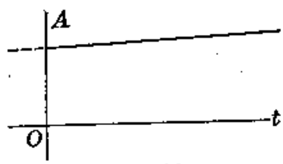

FIG. 29

(*c*) The amount of one dollar at 5% simple interest, after t years, is

$$A = 1 + 0.05t.$$

Any function may be represented graphically by merely plotting the independent variable as abscissa and the function as ordinate. Evidently the graph of a linear function is a straight line, but with this limitation: Due to the nature of the problem, the graph may have a meaning only in a restricted range, or interval, of values of the variable. Thus in (*a*), (*b*) above, we are concerned only with the interval $0 \leqq t \leqq 5$; in (*c*), only with the interval $0 \leqq t$. The graphs of these functions are shown in Figs. 27, 28, 29 respectively.

29. Rate of change. If y is a function of x, then as x changes from any given value x_1 to a new value x_2, y will change from its original value y_1 to a new value y_2.

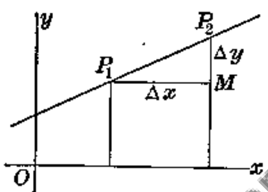

FIG. 30

It will be convenient to denote the *change in x* by the symbol Δx (read delta x), the corresponding *change in y* by the symbol Δy: that is,

$$x_2 - x_1 = \Delta x, \qquad y_2 - y_1 = \Delta y.$$

This notation is commonly used in more advanced mathematics.

If y is a *linear* function of x, it is easily seen that when x changes by any amount, *the change in y is proportional to the change in x.* For, as we pass from any point P_1 to any other point P_2 on the graph, the change in x is P_1M, the change in y is MP_2, and

$$\frac{\Delta y}{\Delta x} = \frac{MP_2}{P_1M} = m,$$

which is constant * — that is, will have the same value no matter what two points P_1, P_2 are chosen.

* To say that any quantity u is *proportional to* another quantity v means that $u = kv$, or $\frac{u}{v} = k$, where k is constant. Here $u = \Delta y$, $v = \Delta x$, $k = m$.

The ratio of the change in y to the change in x is the *rate of change* of the linear function.* We have at once the

THEOREM: *The rate of change of a linear function is constant, and equal to the slope of its graph.*

Further, a positive rate (slope) means that the function is *increasing;* negative rate, *decreasing*. See Figs. 27–29.

EXERCISES

Express the function by a formula and draw the graph, indicating that portion of the graph that has a meaning. Determine the rate of change.

1. The distance covered in time t, by a man running 8 yd. per sec.

2. The value of a consignment of grain at 60¢ per bu., as a function of the number of bushels.

3. The function of Ex. 2, if $100 must be deducted for transportation.

4. The value of a farm at $50 per acre, with buildings worth $2000.

5. The total area (including both bases) of a right circular cylinder of given radius r, as a function of the altitude h.

6. The radius of a circle increases 1 ft. How much does the circumference increase if the original radius is (*a*) 1 inch? (*b*) 1 mile?

In Exs. 7–10, assume that B is 200 mi. east of A.

7. A car, starting at noon, drives east from A at 30 mi. per hr. A second car, starting at 2 : 00 P.M., drives east from A at 50 mi. per hr. Find analytically and graphically (*a*) when and where the cars will be together; (*b*) their relative positions when the faster car reaches B.

8. A car, starting at noon, drives east from A at 40 mi. per hr. A second car, starting at 3 : 00 P.M., drives east from B at 25 mi. per hr. Find analytically and graphically when and where they will be together.

9. A car, starting at noon, drives east from A at 50 mi. per hr. A second car, starting at 1 : 00 P.M., drives west from B at 40 mi. per hr. Find analytically and graphically when and where they will meet.

10. A car, starting at noon, drives east from A at 40 mi. per hr. A second car, starting at 1 : 00 P.M., drives west from B at 30 mi. per hr. A third car, starting at 2 : 00 P.M., drives west from B at 50 mi. per hr. Determine analytically and graphically the period during which the first car will be between the other two.

* Note carefully that *this statement applies to linear functions only.*

11. A manufacturer can turn out 100 units of his product for \$450, 400 units for \$600. Assuming that the cost C is a linear function of the number of units n, determine the function and draw the graph. What is the unit cost? What is the meaning of the C-intercept?

Ans. $C = \frac{1}{2}n + 400.$

12. A railroad can just break even on a certain run, carrying 50 passengers at \$4.50 or 100 at \$2.50. Assuming that the cost is a linear function of the number of passengers, determine the function and explain the meaning of the constants. *Ans.* $C = \frac{1}{2}n + 200.$

13. For the lever of Fig. 31, with the fulcrum at A, it is shown in mechanics that the force necessary to balance the weight W is

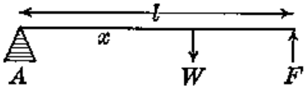

Fig. 31

$$F = \frac{W}{l} \cdot x + \tfrac{1}{2}wl,$$

where w is the weight of the lever per linear foot. For a weight $W = 10$ lbs., it was found that when $x = 8$, $F = 46$; when $x = 16$, $F = 51$. Find l and w, and draw the graph.

Ans. $l = 16$ ft.; $w = 5$ lbs. 2 oz. per ft.

14. For the lever of Fig. 32, with the fulcrum at A, it is shown in mechanics that the force F necessary to balance the weight W is

$$F = \frac{l_1}{l_2} W + \frac{w}{2l_2}(l_1^2 - l_2^2),$$

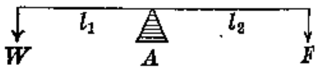

Fig. 32

where w is the weight of the lever per linear foot. With a lever weighing 4 lbs. per ft., it was found that when $W = 20$, $F = 4$; when $W = 30$, $F = 9$. Find l_1 and l_2, and draw the graph. *Ans.* $l_1 = 2$, $l_2 = 4$.

15. Draw a curve from which $\log 2x^n$ may be read if $\log x$ is given.

30. Intercept form. Let the intercepts of a straight line on the axes be $OA = a$ and $OB = b$. Since the line passes through the points $(a, 0)$ and $(0, b)$, its equation may be found at once (§ 24). It is left as an exercise for the student to show that the result is

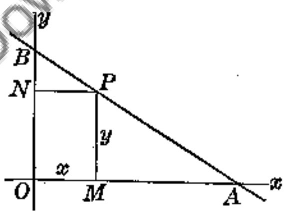

Fig. 33

$$bx + ay = ab.$$

Dividing both members by ab, we get the *intercept form*

$$(1) \qquad \frac{x}{a} + \frac{y}{b} = 1.$$

To write the equation of any line in the intercept form we have merely to *find the intercepts and substitute in* (1). This form evidently fails in case the line passes through the origin or is parallel to either axis.

Example: A line forms with the axes a triangle of area 16 and passes through the point (3, 2). Find the dimensions of the triangle.

Use of the intercept form (1) is suggested by the fact that we can then express the first condition at once:

(2) $$\tfrac{1}{2}ab = 16.$$

Substituting the coördinates (3, 2) in (1), we get

(3) $$\frac{3}{a} + \frac{2}{b} = 1.$$

Equations (2) and (3) give

$$a = 4, \quad b = 8, \qquad \text{or} \qquad a = 12, \quad b = \tfrac{8}{3}.$$

EXERCISES

Draw the following lines; then write the equations, and simplify.

1. Intercepts 4, -1.
2. Intercepts -2, -3.
3. Intercepts $\frac{2}{3}$, $-\frac{1}{2}$.
4. Intercepts $\frac{4}{3}$, $\frac{4}{5}$.

Reduce the following equations to the intercept form. Draw the lines.

5. $3x + 2y = 6$.
6. $5x + 2y = 20$.
7. $2x + 3y + 5 = 0$.
8. $4x - 5y + 7 = 0$.

9. A line makes equal intercepts on the axes and passes through (7, 3). Find its equation in two ways. (§§ 30, 23.)

10. A line makes equal intercepts on the axes and passes through $(-2, 1)$. Find its equation in two ways.

11. The y-intercept is twice the x-intercept; the line passes through $(4, -3)$. Find its equation.

12. The product of the intercepts is 1; the line passes through $(6, -1)$. Find its equation. *Ans.* $x + 4y = 2$; $x + 9y + 3 = 0$.

13. A line passes through $(-4, 1)$ and forms with the axes a triangle of area 1. Find its equation. *Ans.* $x + 2y + 2 = 0$; $x + 8y = 4$.

14. A line passes through $(-7, 2)$, and the segment of the line intercepted between the axes is of length $5\sqrt{2}$. Find its equation.

15. A rectangle is inscribed in a right triangle of base b and height h. What relation must hold between the base and altitude of the rectangle?

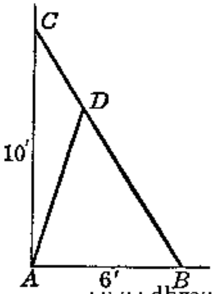

FIG. 34

16. A right circular cylinder is inscribed in a right circular cone of radius r and height h. What relation must hold between the radius and height of the cylinder?

17. A beam BC leans against a wall AC and is stayed by a strut AD. If D is 2 ft. out from the wall, find the length of the strut. *Ans.* 6 ft. 11.5 in.

18. In Ex. 17, if a strut 6 ft. long is used, at what height above the ground will it reach the beam? *Ans.* 5 ft. 3.5 in.

19. Obtain a general solution for the problem of the example: i.e. find the equation of a line forming with the axes a triangle of area A and passing through any point (x_1, y_1). Show that the problem is impossible if $x_1y_1 > \frac{1}{2}A$.

$$Ans.\ a = \frac{A \pm \sqrt{A^2 - 2Ax_1y_1}}{y_1},\ b = \frac{A \mp \sqrt{A^2 - 2Ax_1y_1}}{x_1}.$$

31. The normal form. A straight line is determined if we know the *length p of the perpendicular from the origin* to the line, together with the *angle β which this perpendicular makes with Ox.* To find the equation of a line determined in this way, we assume a point $P : (x, y)$ on the line (the line KPL in Fig. 35) and draw auxiliary lines as shown. Now

$$\begin{aligned} OR &= x\cos\beta, \\ QP &= y\sin\beta, \\ OR + QP &= OR + RN \\ &= ON = p, \end{aligned}$$

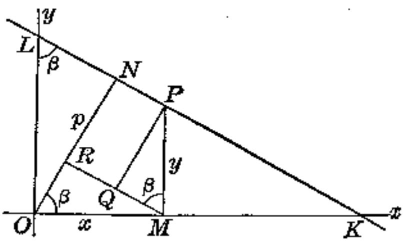

FIG. 35

so that *

$$x\cos\beta + y\sin\beta = p. \qquad (1)$$

* This method of derivation fails in case the given line passes through the origin. But in that case $p = 0$, and the equation of a line through $(0, 0)$ parallel to (1) is obviously $x\cos\beta + y\sin\beta = 0$.

This is the *normal form* of the equation of the straight line. It is chiefly useful in problems involving the distance of a point from a line, distance between parallel lines, etc.

For definiteness we shall adopt the conventions that *p is always positive* and *β is a positive angle less than* 360°.

To reduce the equation of any line

$$Ax + By + C = 0 \tag{2}$$

to the normal form (1), we note that if equations (2) and (1) are to represent the same straight line, the coefficients A, B, C in (2) must be proportional, respectively, to the coefficients $\cos\beta$, $\sin\beta$, $-p$ in (1). Let the constant ratio between these numbers be denoted by k:

$$\frac{\cos\beta}{A} = k, \qquad \frac{\sin\beta}{B} = k, \qquad \frac{-p}{C} = k,$$

so that

$$\cos\beta = kA, \quad \sin\beta = kB, \qquad p = -kC.$$

Square and add the first two of these equations:

$$\cos^2\beta + \sin^2\beta = k^2(A^2 + B^2),$$

whence

$$k^2(A^2 + B^2) = 1, \qquad \text{and} \qquad k = \pm\frac{1}{\sqrt{A^2 + B^2}}.$$

Thus

$$\begin{cases} \cos\beta = \pm\dfrac{A}{\sqrt{A^2 + B^2}}, \qquad \sin\beta = \pm\dfrac{B}{\sqrt{A^2 + B^2}}, \\ \qquad\qquad p = \mp\dfrac{C}{\sqrt{A^2 + B^2}}, \end{cases} \tag{3}$$

where the sign before the radical is to be chosen so that p shall be positive. This leads to the following

RULE: *To reduce the equation of any line $Ax+By+C=0$ to the normal form, divide by $\sqrt{A^2+B^2}$ and choose signs so that the constant term is positive in the right member.**

Example: Find the locus of points at the distance $\sqrt{5}$ from the line $x - 2y + 10 = 0$.

By the rule, the given equation is, in the normal form,

$$-\frac{x}{\sqrt{5}} + \frac{2y}{\sqrt{5}} = \frac{10}{\sqrt{5}},$$

so that the distance of this line from the origin is $p = \frac{10}{\sqrt{5}}$.

The required locus consists of the two lines parallel to, and at a distance $\sqrt{5}$ from, the given line; the distance of these lines from (0, 0) is $\frac{10}{\sqrt{5}} \pm \sqrt{5}$. Hence the required lines are

$$-\frac{x}{\sqrt{5}} + \frac{2y}{\sqrt{5}} = \frac{10}{\sqrt{5}} \pm \sqrt{5}, \quad \text{or} \quad -x + 2y = 10 \pm 5.$$

EXERCISES

Reduce the following equations to the normal form, and find the distance of each line from the origin.

1. $4x + 3y - 7 = 0$.
2. $3x - y - 5 = 0$.
3. $x - 4y + 6 = 0$.
4. $5x + 2y + 5 = 0$.
5. $y = 4 - 5x$.
6. $y = 2x - 3$.
7. $5x - 4y = 0$.
8. $x + 3y = 0$.
9. $y = mx + b$.
10. $y - y_1 = m(x - x_1)$.

Find the equations of the following lines.

11. Parallel to $3x + 4y + 25 = 0$ and passing (*a*) at a distance 2 from the origin; (*b*) 3 units farther from the origin; (*c*) at a distance 1 from the given line. *Ans.* (*b*) $3x + 4y = \pm 40$; (*c*) $3x + 4y + 25 = \pm 5$.

12. Parallel to $x - y = 6$ and passing (*a*) at a distance 6 from the origin; (*b*) $2\sqrt{2}$ units farther from the origin; (*c*) at a distance $\sqrt{2}$ from the given line. *Ans.* (*b*) $x - y = \pm 10$; (*c*) $x - y = 6 \pm 2$.

* When $p = 0$, it is evident that β may have either of two values. This checks with the fact that when $p = 0$, the rule fails to determine the sign.

13. Parallel to $x + 3y + 4 = 0$ and passing at a distance $\sqrt{10}$ from $(-1, 2)$. *Ans.* $x + 3y = 5 \pm 10$.

14. Parallel to $2x - 3y = 6$ and passing at a distance $2\sqrt{13}$ from $(3, 1)$. *Ans.* $2x - 3y = 3 \pm 26$.

15. Parallel to $x - y + 2 = 0$ and passing at a distance 3 from $(2, 3)$.

16. Parallel to $4y = 3x$ and passing at a distance 2 from $(3, 1)$.

17. A circle of radius 3 is tangent to the line $2x - 5y = 2$. Find the locus of its center.

18. A circle of radius 2 is tangent to the line $5x + 12y + 26 = 0$. Find the locus of its center.

19. One side of a square is the line from $(0, 5)$ to $(3, 1)$. Find the other vertices. *Ans.* $(4, 8)$, $(7, 4)$; $(-4, 2)$, $(-1, -2)$.

20. The base of a triangle is the line from $(6, 4)$ to $(5, 2)$; the area is $\frac{5}{2}$. Find the locus of the third vertex. *Ans.* $2x - y = 13$; $2x - y = 3$.

21. In Ex. 20, if the triangle is isosceles, find the third vertex. *Ans.* $(\frac{15}{2}, 2)$; $(\frac{7}{2}, 4)$.

22. The base of a triangle joins the points $(-2, 4)$, $(-1, 3)$; the area is $\frac{9}{2}$. Find the locus of the third vertex.

23. In Fig. 34, p. 44, what is the length of the shortest strut AD that can be used? *Ans.* 5 ft. $1\frac{3}{4}$ in.

24. Solve the problem of Ex. 23 in general: i.e. in Fig. 34, take $AB = a$, $AC = b$. *Ans.* $\dfrac{ab}{\sqrt{a^2 + b^2}}$.

Find the distance between the given lines.

25. $3x - y = 6, \quad 3x - y = 8$. *Ans.* $\frac{1}{5}\sqrt{10}$.

26. $x + 2y + 5 = 0, \quad x + 2y - 3 = 0$. *Ans.* $\frac{8}{5}\sqrt{5}$.

27. $x + 4y - 1 = 0, \quad x + 4y + 6 = 0$.

28. $2x - 3y + 2 = 0, \quad 2x - 3y + 3 = 0$.

29. Two sides of a square lie in the lines $5y = 3x + 4$, $3x - 5y = 5$. Find the area of the square. *Ans.* $\frac{81}{34}$.

30. Find the area of the rectangle bounded by the lines $x - 2y = 3$, $2x + y + 2 = 0$, $2x + y + 4 = 0$, $x - 2y + 4 = 0$.

31. A circle is tangent to the lines $4x + y + 3 = 0$, $4x + y - 7 = 0$. Find its area, and the locus of its center.

32. In Fig. 35, what are the coördinates of N?

33. Derive the normal form from the fact that $\overline{ON}^2 + \overline{NP}^2 = \overline{OP}^2$.

34. Derive the normal form from the fact that ON is perpendicular to NP. (§ 9.)

32. Distance of a point from a line. To find the distance from the line

$$Ax + By + C = 0$$

(the line λ in Fig. 36) to any point $P_1 : (x_1, y_1)$ not on that line, let us suppose the equation reduced to the normal form:

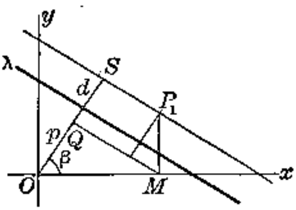

Fig. 36

$$x \cos \beta + y \sin \beta = p.$$

Then, drawing auxiliary lines as shown, we see that

$$OQ = OM \cos \beta = x_1 \cos \beta,$$
$$QS = MP_1 \sin \beta = y_1 \sin \beta,$$
$$OS = x_1 \cos \beta + y_1 \sin \beta.$$

Since $d = OS - p$, we have

$$d = x_1 \cos \beta + y_1 \sin \beta - p.$$

Upon substituting in this formula the values of $\cos \beta$, $\sin \beta$, and p given by (3), § 31, we obtain the following:

The distance from the line

$$Ax + By + C = 0$$

to the point (x_1, y_1) *is given by the formula*

(1) $$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}},$$

where the sign before the radical is to be chosen opposite to the sign of C.

It appears from Fig. 36 that the distance as given by the formula will be positive if the point and the origin are on opposite sides of the line, negative if they are on the same side. Usually, however, we are concerned chiefly with the numerical value apart from sign.

Examples: (*a*) Find the distance from the straight line $3x - 2y + 5 = 0$ to the point (3, 4).

Formula (1) gives at once

$$d = \frac{3 \cdot 3 - 2 \cdot 4 + 5}{-\sqrt{13}} = -\frac{6}{\sqrt{13}}.$$

(*b*) Find the equations of the lines bisecting the angles between the lines

$$x + y = 2, \qquad x - 7y + 2 = 0$$

(the lines λ_1, λ_2 of Fig. 37).

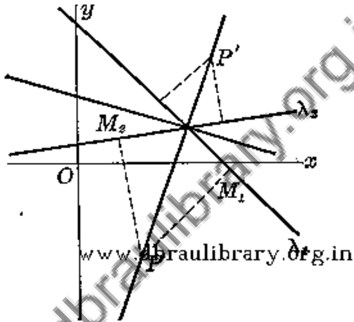

FIG. 37

The bisector of the angle between two lines is the locus of points *equidistant from the two lines.* Assume a point $P : (x, y)$ in the bisector of that angle in which the origin lies. Then the distances M_1P and M_2P are numerically equal; they are also algebraically equal, since in both cases O and P are on the same side of the line and both distances are negative, or when P is in the position P' both are positive. But, by (1),

$$M_1P = \frac{x + y - 2}{\sqrt{2}}, \qquad M_2P = \frac{-x + 7y - 2}{5\sqrt{2}},$$

so that the equation of the locus of P is

$$\frac{x + y - 2}{\sqrt{2}} = \frac{-x + 7y - 2}{5\sqrt{2}},$$

or

$$3x - y = 4.$$

In a similar way the equation of the other angle bisector is found to be

$$\frac{x + y - 2}{\sqrt{2}} = -\left(\frac{-x + 7y - 2}{5\sqrt{2}}\right),$$

or

$$x + 3y = 3.$$

EXERCISES

Find the distance of the point from the line.

1. $(5, 2)$, $3x - y + 6 = 0$. *Ans.* $-\frac{19}{10}\sqrt{10}$.

2. $(3, -5)$, $2x + y + 4 = 0$. *Ans.* $-\sqrt{5}$.

3. $(4, 3)$, $3x + 2y = 6$. **4.** $(0, 5)$, $2y = x + 6$.

5. $(3, -1)$, $x - 4y = 0$. **6.** $(-2, 0)$, $3x + y = 0$.

7. A moving point is equidistant from the origin and the line $x + y = 2$. Find the equation of its locus.

Ans. $x^2 - 2xy + y^2 + 4x + 4y - 4 = 0$.

8. A circle passes through $(0, 0)$ and is tangent to the line $2x+3y=1$. Find the locus of its center. *Ans.* $9x^2 - 12xy + 4y^2 + 4x + 6y = 1$.

9. A circle passes through $(1, 1)$ and is tangent to the line $3x-y=6$. Find the locus of its center.

Ans. $x^2 + 6xy + 9y^2 + 16x - 32y - 16 = 0$.

10. A moving point is twice as far from the origin as from the line $x - y = 1$. Find the equation of its locus.

Ans. $x^2 - 4xy + y^2 - 4x + 4y + 2 = 0$.

Find the bisectors of the angles between the given lines.

11. $13x - 9y = 10$, $x + 3y = 6$.

Ans. $2x - 6y + 5 = 0$; $9x + 3y = 20$.

12. $2x + y + 1 = 0$, $11x + 2y = 2$.

Ans. $21x + 7y + 3 = 0$; $x - 3y = 7$.

13. $x = 2$, $y = \frac{3}{4}x$. **14.** $x = 3$, $y = 5$.

15. $3x - y = 2$, $y = 3x + 4$. **16.** $y = 2x$, $2x - y = 6$.

17. $y = 3x$, $x + y = 1$. *Ans.* $(3 \pm \sqrt{5})x - (1 \mp \sqrt{5})y = \pm\sqrt{5}$.

18. A circle is tangent to the x-axis and the line $5y = 12x$. Find the locus of its center. *Ans.* $2x - 3y = 0$; $3x + 2y = 0$.

19. A point moves so that its distance from the line $x + y = 3$ is 4 times its distance from the line $7x - y + 1 = 0$. Find the equation of its locus. *Ans.* $33x + y = 11$; $23x - 9y + 19 = 0$.

20. A point moves so that the ratio of its distances from two fixed lines is constant. Prove analytically that its locus is two straight lines.

21. A point moves so that the square of its distance from the point $(1, 0)$ is numerically equal to its distance from the line $x = 1$. Find the equation of its locus. *Ans.* $x^2 + y^2 - x = 0$; $x^2 + y^2 - 3x + 2 = 0$.

22. A point moves so that the product of its distances from two parallel lines is constant. Show that its locus consists of two lines parallel to the given lines.

CHAPTER V

THE CIRCLE

33. Definitions; standard forms. A *circle* is the locus of a point that moves at a constant distance from a fixed point. The fixed point is the *center*, and the constant distance is the *radius*. The radius as thus defined is of course merely a number of linear units; the term is also used, as in elementary geometry, to mean a line-segment joining the center and a point of the curve. A *diameter* of a circle may mean either a straight line through the center, or the segment of such a line lying inside the curve.

For the circle with center at O and radius a, we have for any position of the moving point

$$\sqrt{x^2 + y^2} = a;$$

if the center is at any point $C:(h, k)$, as in Fig. 38,

$$\sqrt{(x - h)^2 + (y - k)^2} = a.$$

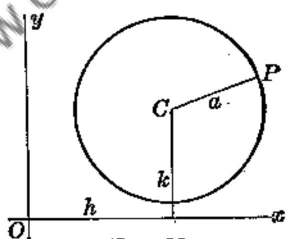

FIG. 38

Hence: *The equation of a circle of radius a is, if the center is the point* (0, 0),

$$(1) \qquad x^2 + y^2 = a^2;$$

if the center is the point (h, k),

$$(2) \qquad (x - h)^2 + (y - k)^2 = a^2.$$

Equations (1) and (2) are called *standard forms* of the equation of the circle. Of course (1) is merely a special case of (2): the case $h = k = 0$.

34. General equation. It appears from (2), § 33, that the equation of a circle is always of the second degree.

The most general equation of the second degree in x and y may contain, at most, terms in x^2, xy, y^2, x, y, and a constant: i.e. it may be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Consider now the special case in which $A = C$ and $B = 0$:

$$(1) \qquad Ax^2 + Ay^2 + Dx + Ey + F = 0.$$

We may always divide this equation through* by A, transpose the constant term to the right member, and complete the squares in x and y; the equation then has the form

$$(2) \qquad (x - h)^2 + (y - k)^2 = a^2,$$

and consequently represents a circle whenever the right member is positive. Equation (1) is the *general form* of the equation of the circle.

Conversely, it appears from § 33 that the equation of every circle may be put in the form (1).

When an equation of form (1) is reduced to form (2), it may happen that the right member becomes 0:

$$(x - h)^2 + (y - k)^2 = 0.$$

Since this equation holds only when $x = h$ and $y = k$, the locus is the single point (h, k) — a so-called "point-circle."

Finally, it may happen that the right member of (2) is negative: in this case there is clearly no locus.

Therefore, we have the

THEOREM: *An equation of the second degree in which x^2 and y^2 have equal coefficients and the xy-term is missing represents a circle (exceptionally, a single point, or no locus); and conversely.*†

* If A were 0, we should no longer have an equation of second degree.
† See the footnote, p. 36.

Example: Find the center and radius of the circle

$$4x^2 + 4y^2 - 4x + 2y + 1 = 0.$$

First transpose the constant term to the right member and divide by 4:

$$x^2 + y^2 - x + \tfrac{1}{2}y = -\tfrac{1}{4}.$$

Then complete the squares in x and y:

$$x^2 - x + \tfrac{1}{4} + y^2 + \tfrac{1}{2}y + \tfrac{1}{16} = -\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{16},$$

or

$$(x - \tfrac{1}{2})^2 + (y + \tfrac{1}{4})^2 = \tfrac{1}{16}.$$

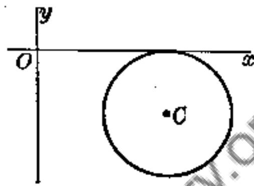

FIG. 39

The center is the point $C : (\frac{1}{2}, -\frac{1}{4})$, and the radius is $\frac{1}{4}$.

EXERCISES

Write the equations of the following circles.

1. With center at $(-5, 3)$, radius $\sqrt{2}$.
2. With center at $(3, -2)$, radius $\sqrt{13}$.
3. With center at $(a, 2a)$, and tangent to the x-axis.
4. With radius a and tangent to both axes.
5. With center at $(5, -2)$ and passing through $(4, 3)$.
6. With center at $(-1, -2)$ and passing through $(0, 2)$.
7. With center at $(0, 0)$ and tangent to the line $2x - y = 5$.
8. With center at $(2, 3)$ and tangent to the line $x - 3y + 4 = 0$.
9. With center at $(-1, 2)$ and tangent to the line $4x - y = 5$.
10. Using equation (2), § 33, find the condition that a circle shall
    (*a*) be tangent to the y-axis; (*b*) be tangent to both axes; (*c*) pass through $(0, 0)$; (*d*) have its center on Ox; (*e*) have its center on the line $y = mx + b$.

Draw the following circles.

11. $x^2 + y^2 - 6x + 10y - 2 = 0$.
12. $x^2 + y^2 + 2x + 8y + 8 = 0$.
13. $x^2 + y^2 = 2ax$.
14. $x^2 + y^2 = 3y$.
15. $x^2 + y^2 - 6x + 8y = 0$.
16. $x^2 + y^2 + 10x - 24y = 0$.
17. $3x^2 + 3y^2 + 2x - 4y = 0$.
18. $2x^2 + 2y^2 - 3x - 5y + 3 = 0$.

19. Prove that equation (1), § 34, represents a point-circle if and only if $D^2 + E^2 = 4AF$.

20. Prove that (1), § 34, has no locus if and only if $D^2 + E^2 < 4AF$.

21. A point moves so that the sum of the squares of its distances from the points $(a, 0)$, $(-a, 0)$ is constant (equal to k^2). Find the equation of its locus, and draw the curve for the cases $k^2 = 6a^2$, $k^2 = 4a^2$, $k^2 = 3a^2$, $k^2 = 2a^2$, $k^2 = a^2$.

22. Let A, B be two fixed points. A point P moves so that $\overline{AP}^2 + k \cdot \overline{BP}^2 = l^2$, where k, l are constant. What kind of curve is described? Take A, B as $(a, 0)$, $(-a, 0)$. Is the proof general? Discuss special cases.

23. A point moves so that the square of its distance from a fixed point is a constant multiple of its distance from a fixed line. What kind of curve is described? Discuss all cases.

24. Taking A, B as in Ex. 22, find the locus of a point P moving so that $AP = k \cdot PB$. Draw the curves $k = \frac{1}{2}$, $k = \frac{9}{10}$, $k = \frac{1}{10}$. What happens as k approaches 0? Approaches 1? Increases indefinitely?

25. Prove analytically that an angle inscribed in a semicircle is a right angle.

26. Prove that the circles $x^2 + y^2 + 4y = 4$, $x^2 + y^2 = 2x + 10y - 8$ are tangent. Draw the figure.

27. Prove that the circles $x^2 + y^2 - 10x - 8y = 4$, $x^2 + y^2 = 2x + 4y$ are tangent. Draw the figure.

28. On the line $x + 2y = 7$, find points at a distance 5 from $(3, 7)$.

Ans. $(3, 2)$, $(-1, 4)$.

29. On the circle $x^2 + y^2 - 4x - 10y + 19 = 0$, find points at a distance 5 from $(5, -1)$.

Ans. $(1, 2)$, $(5, 4)$.

30. Find the intersections of the circles $x^2 + y^2 - 2x - 4y - 3 = 0$, $x^2 + y^2 - 4x - 6y - 5 = 0$. What does the result prove?

35. Number of points required to determine a curve. We know from elementary geometry that a circle is determined by three points. This is proved analytically by the fact that the equation

$$(x - h)^2 + (y - k)^2 = a^2 \qquad (1)$$

contains three independent constants, h, k, and a. For, when the coördinates of the three points in turn are substituted in (1), we obtain three simultaneous equations to determine h, k, a.

More generally, a circle may be made to satisfy any three conditions which when expressed analytically lead to three simultaneous equations for determining the constants. Thus three tangents may be given, or two points and the radius, etc. It may not, however, be *uniquely* determined; there may be two or more circles satisfying the given conditions. (For instance, see Exs. 11–16 below.)

The above is merely a special instance of the general

THEOREM: *The number of points required to determine a curve is equal to the number of independent constants in the equation of the curve.*

We have had a previous illustration of this fact in the case of the straight line, which as we know is determined by two points, corresponding to the fact that the linear equation contains two constants.*

36. Circle determined by three conditions. The general method of finding the equation of a circle satisfying three conditions is, as suggested in § 35, to express the conditions analytically by means of three simultaneous equations which may be solved for the constants. The conditions may be expressed most simply in some cases by using the standard form (2) of § 33, in other cases by using the general form (1) of § 34. Very often, however, special methods may be devised which are shorter than the general method and are more easily interpreted geometrically. Such methods can usually be found by recalling the construction of elementary geometry for the center and radius.

* It might seem at first thought that the equation $Ax + By + C = 0$ contains three constants. But if we divide through by any one of them, say A, there remain only two constants, $\frac{B}{A}$ and $\frac{C}{A}$. It is clear that the essential constants are not the coefficients, but the *ratios* of any two of them to the third. For example, the equations $y = 2x - 3$, $4x - 2y = 6$, $6y = 12x - 18$ all represent the same line. Similar remarks apply to (1), § 34.

Example: Find the equation of the circle through the points $P_1 : (1, 1)$, $P_2 : (2, -1)$, and $P_3 : (2, 3)$.

One method is to assume the equation of the circle in the form

$$(1) \qquad x^2 + y^2 + Dx + Ey + F = 0.$$

(It is convenient to take $A = 1$.) Substituting the coordinates of the given points in turn, we get

$$\begin{aligned} &(2) & 2 + D + E + F &= 0, \\ &(3) & 5 + 2D - E + F &= 0, \\ &(4) & 13 + 2D + 3E + F &= 0; \end{aligned}$$

these equations may be solved for D, E, and F, and the results substituted in (1). (Or D, E, F may be eliminated from equations (1), (2), (3), (4) directly — see Ex. 24 below.)

The same problem may be solved by the following method, which has a more obvious geometric interpretation. The center lies on the perpendicular bisector of P_1P_2, whose equation is found by familiar methods (cf. example (*a*), §20) to be

$$2x - 4y = 3;$$

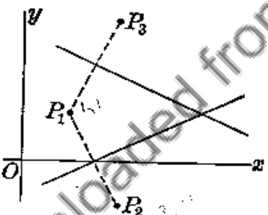

FIG. 40

it also lies on the perpendicular bisector of P_1P_3, whose equation is

$$2x + 4y = 11.$$

The center is therefore the point of intersection of these lines, viz. $(\frac{7}{2}, 1)$. The radius is the distance from $(\frac{7}{2}, 1)$ to any one of the given points, which is found to be $\frac{5}{2}$. By formula (2) of § 33, the equation of the circle is

$$(x - \tfrac{7}{2})^2 + (y - 1)^2 = \tfrac{25}{4}.$$

Evidently this method merely carries out analytically the elementary geometric construction for the center.

EXERCISES

Find the equations of the following circles. Check the answers.

1. Through $(-1, 2)$, $(3, 4)$, $(2, 3)$. Solve by two methods.

2. Through $(5, 3)$, $(3, 1)$, $(-3, -1)$. Solve by two methods.

3. Circumscribing the triangle $(4, 4)$, $(2, 1)$, $(-1, 3)$.

4. Circumscribing the triangle $(4, 3)$, $(2, 3)$, $(3, 6)$.

5. Passing through the points $(-1, -3)$, $(-5, 3)$, and having its center on the line $x - 2y + 2 = 0$. *Ans.* $(x + 6)^2 + (y + 2)^2 = 26$.

6. Passing through the points $(-1, 1)$, $(-7, 3)$, and having its center on the line $2x + y = 9$. *Ans.* $(x + 1)^2 + (y - 11)^2 = 100$.

7. Tangent to the line $4x - 3y = 26$ at $(5, -2)$ and passing through $(-2, -3)$. *Ans.* $x^2 + y^2 - 2x - 2y - 23 = 0$.

8. Tangent to the line $2x + y = 8$ at $(4, 0)$ and passing through $(7, 3)$.

9. Tangent to the line $y = 3x$ at $(1, 3)$ and passing through $(5, 7)$.

10. Tangent to $x + y = 1$ at $(4, -3)$ and passing through $(6, 7)$.

11. Tangent to the x-axis and passing through the points $(4, -1)$, $(-3, -2)$. *Ans.* Centers: $(1, -5)$, $(21, -145)$.

12. Tangent to the y-axis and passing through the points $(1, 5)$, $(8, 12)$. *Ans.* Centers: $(13, 0)$, $(5, 8)$.

13. Tangent to the line $x + y = 4$ at $(1, 3)$, and having a radius $\sqrt{2}$. Solve in two ways. *Ans.* Centers: $(0, 2)$, $(2, 4)$.

14. Tangent to the line $x - 2y = 3$ at $(-1, -2)$, and having a radius $\sqrt{5}$. Solve in two ways. *Ans.* Centers: $(0, -4)$, $(-2, 0)$.

15. Tangent to the lines $x = 3y$, $3x + y = 2$, and having its center on the line $x + y = 3$. *Ans.* Centers: $(5, -2)$, $(\frac{4}{3}, \frac{5}{3})$.

16. Tangent to the lines $x + y = 1$, $7x - y = 5$, and having its center on the line $x + y = 4$. *Ans.* Centers: $(-\frac{3}{4}, \frac{19}{4})$, $(3, 1)$.

17. Having a radius $\sqrt{10}$, through $(0, 3)$, $(2, 5)$.

18. Ex. 17 by a second method.

19. Having a radius $\frac{5}{2}\sqrt{2}$, through $(1, 0)$, $(3, 6)$. Solve in two ways.

20. Inscribed in the triangle formed by the lines $3x - 4y = 5$, $4x - 3y + 10 = 0$, $y = 2$. *Ans.* Center: $(-\frac{5}{21}, \frac{10}{21})$.

21. Tangent to the lines $x + 2y = 4$, $x + 2y = 2$, $y = 2x - 5$. *Ans.* Centers: $(3, 0)$, $(\frac{11}{5}, \frac{2}{5})$.

22. Tangent to the circle $x^2 + y^2 = 100$ at $(6, -8)$ and having a radius 15. *Ans.* Centers: $(-3, 4)$, $(15, -20)$.

23. Tangent to the circle $x^2 + y^2 + 2x - 6y + 5 = 0$ at $(1, 2)$ and passing through $(4, -1)$. *Ans.* Center: $(3, 1)$.

24. Prove that the equation

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

represents the circle through (x_1, y_1), (x_2, y_2), (x_3, y_3).

25. Solve Exs. 1–4 by the formula of Ex. 24.

26. When does the equation of Ex. 24 reduce to an equation of first degree? Interpret geometrically.

37. Curves through the intersections of two given curves. Let there be given any two curves, whose equations we will denote by

(1) $$u = 0, \qquad v = 0,$$

where u and v represent certain expressions* involving x and y. Suppose these curves intersect in certain points P_1, P_2, etc.

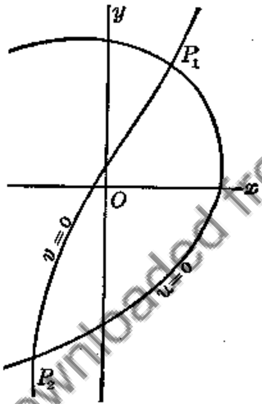

FIG. 41

We will now consider the locus

(2) $$u + kv = 0,$$

where k is an arbitrary constant, and proceed to show that, for all values of k, this curve passes through the points of intersection of the two given curves. Since P_1 lies on the curve $u = 0$, its coördinates when substituted for x and y in the expression u, no matter where u is found, will make that quantity identically zero. Similarly, since P_1 lies on the curve $v = 0$, its coördinates will make v vanish. Therefore, substituting the coördinates of P_1 in (2), we get

$$0 + k \cdot 0 = 0,$$

* The student should make sure that he understands the meaning of the abbreviated notation here used. The letters u and v stand for *any expressions in x and y whatsoever,* not necessarily of the first or second degree.

which is true for all values of k; hence P_1 lies on the curve (2). By the same reasoning this curve passes through all the other points of intersection of the given curves.

In summary, we have proved the

THEOREM: *If $u = 0$ and $v = 0$ are two intersecting curves, the equation*

(3) $$u + kv = 0,$$

where k is an arbitrary constant, represents for any value of k a curve passing through the points of intersection of the given curves.

Since equation (3) contains one undetermined constant, the curve may be made to satisfy one condition (§ 35) — for instance to pass through a given point or to be tangent to a given line.

When $u = 0$, $v = 0$ are two circles, so that u and v stand for two expressions of the form

$$A_1x^2 + A_1y^2 + D_1x + E_1y + F_1,$$
$$A_2x^2 + A_2y^2 + D_2x + E_2y + F_2,$$

it is easily seen that in general the equation

(4) $$u + kv = 0$$

is of the form (1), § 34, and therefore represents a circle.*

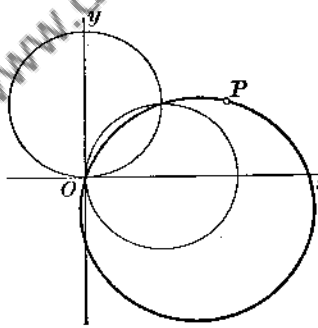

FIG. 42

Since this circle must pass through the two points of intersection of the given circles, the fact that the new circle is determined by one more condition appears independently of the constant-counting theorem of § 35.

* There is one exceptional case; in Ex. 17 below the student is asked to discover this case.

It should also be noted that even when the given circles do not intersect, equation (4) will still be of form (1), § 34 (except as just remarked), and therefore will in general represent a circle.

Example: Find the circle through the points of intersection of the circles $x^2 + y^2 = y$, $x^2 + y^2 = x$ and through the point $P : (1, \frac{1}{2})$. (See Fig. 42.)

By (4), the required equation will have the form

(5) $$x^2 + y^2 - y + k(x^2 + y^2 - x) = 0.$$

If this circle is to pass through $(1, \frac{1}{2})$, these coördinates may be substituted for x and y:

$$1 + \tfrac{1}{4} - \tfrac{1}{2} + k(1 + \tfrac{1}{4} - 1) = 0,$$

whence

$$k = - 3.$$

Substituting this value of k in (5), we get

$$x^2 + y^2 - y - 3(x^2 + y^2 - x) = 0,$$

or

$$2x^2 + 2y^2 - 3x + y = 0.$$

EXERCISES

Find the equations of the following straight lines.

1. Through the intersection of the lines $3x - y + 2 = 0$, $x + 2y = 3$ and through (1, 2). *Ans.* $8y = 3x + 13$.

2. Through the intersection of the lines $2x + 3y + 1 = 0$, $6x - 5y = 3$ and through $(-2, -2)$. *Ans.* $11x - 15y = 8$.

3. Through the intersection of the lines of Ex. 2, and parallel to Ox.

4. Through the intersection of the lines of Ex. 2, and parallel to Oy.

5. Through the intersection of the lines of Ex. 2, and having the slope 2. *Ans.* $14x - 7y = 5$.

6. Through the intersection of the lines $x - 3y = 6$, $4x + y = 0$, and having the slope -3. *Ans.* $39x + 13y + 6 = 0$.

7. What is represented by equation (3), § 37, when the original curves are two parallel lines?

8. If $u = 0$ represents a circle and $v = 0$ represents a line, what is the locus of the equation $u + kv = 0$?

Find the equations of the following circles.

9. Through the intersections of the circles $x^2 + y^2 - x + y = 2$, $x^2 + y^2 = 5$, and through $(2, -2)$. *Ans.* $x^2 + y^2 - 3x + 3y + 4 = 0$.

10. Through the intersections of the circles $x^2 + y^2 = 6x - 2y - 1$, $x^2 + y^2 = 4x$, and through (1, 3). *Ans.* $5x^2 + 5y^2 - 8x - 12y - 6 = 0$.

11. Through the intersections of the circle $x^2 + y^2 = 25$ and the line $2x - 3y = 5$, and through the point (6, 2).

12. Through the intersections of the circle $x^2 + y^2 + 2x + 2y = 0$ and the line $y = x - 1$, and through the point (3, 1).

13. Through the intersections of the circles $x^2 + y^2 + 3x - y = 5$, $2x^2 + 2y^2 - 3x + 2y = 4$, and having its center on the y-axis.

14. Through the intersections of the circles $x^2 + y^2 + x - 2y = 6$, $x^2 + y^2 - 2x + y = 10$, and having its center on the x-axis.

15. Through the intersections of the circles $x^2 + y^2 = 2x$, $x^2 + y^2 = 2y$, and having a radius $\sqrt{5}$.

Ans. $x^2 + y^2 - 4x + 2y = 0$; $x^2 + y^2 + 2x - 4y = 0$.

16. Through the intersections of the circles of Ex. 15, and having its center on the line $y = x$. *Ans.* $x^2 + y^2 = x + y$.

17. If $u = 0$, $v = 0$ are two non-concentric circles, whether intersecting or not, prove that $u + kv = 0$ is also, in general, a circle. Are there any exceptions?

18. What is represented by $u + kv = 0$ when $u = 0$, $v = 0$ are two concentric circles?

19. If two circles $u = 0$, $v = 0$ are tangent to each other at a point P, prove that $u + kv = 0$ represents, in general, a circle tangent to the given circles at P. Is there any exception?

20. Prove geometrically that, when $u = 0$, $v = 0$ are two non-concentric circles, the center of the circle $u + kv = 0$ is on the line of centers of the given circles. Can the proof be carried out in all cases?

21. Solve Ex. 20 analytically. (For convenience, take the given circles with centers on Ox. Is the proof still general?) What is the advantage of the analytic over the geometric proof?

22. Prove that the curve $u + kv = 0$ passes through no point of either $u = 0$ or $v = 0$ except the points of intersection.

23. In the equation $u + kv = 0$, what happens if we substitute the coördinates of a point on the curve $u = 0$? On the curve $v = 0$?

38. Radical axis; common chord. Given two circles

$$u = 0, \qquad v = 0,$$

we can make the coefficients of x^2 and y^2 the same in the two equations if they are not already so. If then we subtract one equation from the other, member by member, the terms in x^2 and y^2 drop out, so that the equation

$$u - v = 0 \tag{1}$$

is of first degree and represents a straight line.* This line is called the *radical axis* of the two circles.

If the circles intersect in distinct points, it follows from the theorem of § 37 that the line (1) is the *common chord.* If the circles are tangent to each other, this line is the *common tangent.*

EXERCISES

1. Find (*a*) the equation of the common chord of the circles $x^2 + y^2 - 6x - 8y = 0$, $x^2 + y^2 = 9$; (*b*) the length of the chord.

Ans. (*b*) $\frac{3}{5}\sqrt{91}$.

2. Find (*a*) the equation of the common chord of the circles $2x^2 + 2y^2 = x$, $x^2 + y^2 + 4x - 2y = 0$; (*b*) the length of the chord.

Ans. (*b*) $\frac{2}{97}\sqrt{97}$.

3. Prove analytically that, in general, *the radical axes of three circles taken in pairs meet in a point* (called the *radical center*).

4. Find the radical center (Ex. 3) of the circles $x^2 + y^2 = 4$, $x^2 + y^2 = 4y$, $x^2 + y^2 - 6x + 8y + 24 = 0$. Draw the figure.

Ans. (6, 1).

5. Find the radical center (Ex. 3) of the circles $x^2 + y^2 + 3x - 2y = 4$, $x^2 + y^2 - 2x - y = 6$, $x^2 + y^2 = 1$. Draw the figure. *Ans.* $(-1, -3)$.

6. Give a geometric construction for the radical axis of two non-intersecting circles, based on the theorem of Ex. 3.

7. Prove analytically that the radical axis of two circles is perpendicular to their line of centers. Hence vary the construction of Ex. 6.

8. In what case do two circles have no radical axis?

9. When do three non-concentric circles have no radical center?

10. Find the radical axis of two point-circles (i.e. circles of radius 0).

11. Prove that the perpendicular bisectors of the sides of a triangle meet in a point. (Exs. 3, 10.)

* There is one exceptional case — see Ex. 8 below

CHAPTER VI

THE CONIC SECTIONS

39. Definitions. The path of a point which moves so that *its distance from a fixed point is in a constant ratio to its distance from a fixed line* is called a *conic section*, or simply a *conic*. (See also § 40.)

The fixed point is the *focus* of the conic, the fixed line the *directrix*, and the constant ratio the *eccentricity*. In Fig. 43, if F is the focus, λ the directrix, and P a point on the conic, then, by the definition,

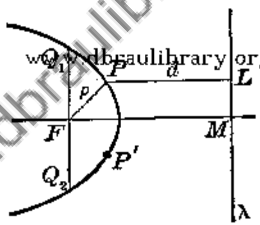

FIG. 43

$$\frac{FP}{LP} = e,$$

or

(1) $$\rho = ed,$$

where e denotes the eccentricity.

Equation (1) would still be true if the point P were in the position P', symmetric to P with respect to the line FM. Hence the line through the focus perpendicular to the directrix is an axis of symmetry for the curve. It follows that the line through the focus parallel to the directrix intersects the curve in two points; the chord Q_1Q_2 joining these points is the *latus rectum*.

The conic sections fall into three classes, differing greatly in form and in certain of their properties, and distinguished by the value of e, as follows:

if $e < 1$, the conic is an *ellipse* * (Fig. 50, p. 74);
if $e = 1$, the conic is a *parabola* (Fig. 44, p. 65);
if $e > 1$, the conic is a *hyperbola* * (Fig. 53, p. 79).

*For alternative definitions of the ellipse and hyperbola, see §§ 46, 52.

40. The circle; degenerate conics. In addition to the three typical "conics" defined above, it will be convenient to include under that term various other loci, some of which do not satisfy the definition of § 39.

First, when $e = 0$ the definition fails. But when e *approaches* 0, the ellipse becomes more and more nearly circular; we will therefore agree to consider the circle as a limiting case of the ellipse.

Second, for reasons both geometric and analytic, we will agree to consider the "point-ellipse" (§ 48), two parallel or coincident lines (§ 43), and two intersecting lines (§ 54) as conic sections of exceptional type. These loci are sometimes called *degenerate conics*.

It can be shown that every plane section of a right circular cone is a curve of the class that we have called "conic sections" (provided the forms just discussed are included); this of course is the reason for the name. The student may amuse himself by discovering, intuitively, how the cutting plane must be passed in order to obtain the various sections.*

We will also establish in due course the important

THEOREM: *An equation of the second degree represents a conic section (exceptionally, no locus); and conversely.*†

The property embodied in the theorem is sometimes used as a definition, instead of the one given in § 39:

A *conic section* is a curve whose equation is of the second degree.

This definition automatically includes not merely the three typical conics, but also the exceptional forms listed above.

* Two parallel lines obviously cannot be cut from a cone. They can, however, be cut from a cylinder, which is the form approached by the cone as the vertex recedes indefinitely.

† See the footnote, p. 36.

I. The Parabola

41. First standard form. The parabola has been defined in § 39 as *the conic whose eccentricity is* 1; that is,

The parabola is the locus of points which are equidistant from a fixed point and a fixed line.

The line through the focus perpendicular to the directrix is called the *axis* of the curve. The point where the axis intersects the curve, i.e. the point midway between the focus and the directrix, is the *vertex* of the parabola. The (undirected) distance from the vertex to the focus will be denoted by the letter a, so that a is always positive.

Let us take the vertex of a parabola as the origin and the focus at $(a, 0)$, where a is any *positive* number. Then the axis of the curve is the x-axis and the directrix is the line $x = -a$. If $P: (x, y)$ is any point on the curve, then by the definition of the parabola

$$\sqrt{(x-a)^2 + y^2} = a + x,$$

which reduces to

$$y^2 = 4ax. \tag{1}$$

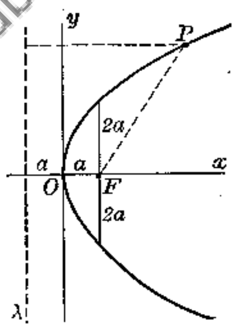

Fig. 44

From (1) it appears that x must be positive, otherwise y^2 would be negative and y imaginary; hence the curve lies entirely to the right of the y-axis. For every positive value of x there are two values of y, numerically equal but of opposite sign, which increase numerically as x increases: thus the curve opens indefinitely to the right.

When $x = a$, $y = \pm 2a$. Hence the *length of the latus rectum is four times the distance between the vertex and focus.* This property, being intrinsic for the parabola, is independent of the position of the curve with reference to the coördinate axes.

42. Other standard forms. The equation $y^2 = 4ax$ expresses a characteristic geometric property of the parabola — a property true for all parabolas, regardless of their position with reference to the coördinate axes. Given a parabola with vertex V and focus F (Fig. 45), let us drop from P a perpendicular MP to the axis of the curve. Then by (1), § 41,

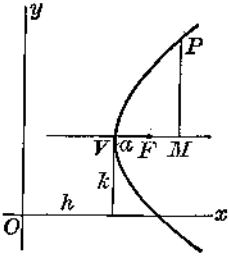

FIG. 45

(1) $$\overline{MP}^2 = 4VF \cdot VM;$$

this is the characteristic property just mentioned.

Consider now a parabola with vertex at (h, k) (Fig. 45), axis parallel to Ox, and focus at a distance a to the right of the vertex. Choosing a point $P : (x, y)$ on the curve, we see by (1) that the equation of the parabola is

$$(y - k)^2 = 4a(x - h).$$

By similar argument we may obtain the equation when the curve opens to the left, or when it opens upward or downward (axis parallel to Oy). In summary, we have the following results:

The equation of a parabola with vertex at (h, k) is, if the axis is parallel to Ox and the curve opens to the right,

(2) $$(y - k)^2 = 4a(x - h);$$

if the axis is parallel to Ox and the curve opens to the left,

(3) $$(y - k)^2 = - 4a(x - h);$$

if the axis is parallel to Oy and the curve opens upward,

(4) $$(x - h)^2 = 4a(y - k);$$

if the axis is parallel to Oy and the curve opens downward,

(5) $$(x - h)^2 = - 4a(y - k).$$

EXERCISES

In the following cases, locate the vertex, the ends of the latus rectum, and a few other points, and trace the curve.

1. $y^2 = -8x$.
2. $y^2 = 3x$.
3. $x^2 = 6y$.
4. $x^2 + y = 0$.
5. $4y^2 + x = 0$.
6. $8x^2 = 3y$.
7. $(x + \frac{1}{2})^2 = \frac{4}{3}(y - 1)$.
8. $(y - 2)^2 = -12(x + 3)$.
9. $(y + 3)^2 = -2(x + 1)$.
10. $(x - \frac{1}{4})^2 = \frac{1}{2}y$.
11. $(x - 10)^2 = 100(y - 5)$.
12. $(y + 6)^2 = 6(x - 8)$.

Find the equations of the following parabolas.

13. With vertex at O, axis Ox, and passing through $(3, -2)$.

14. With vertex at O, axis Oy, and passing through $(-1, 4)$.

15. With vertex $(3, -1)$ and focus $(3, -2)$.

16. With vertex $(-1, -2)$ and focus $(-4, -2)$.

17. With vertex $(2, 4)$ and directrix $x = 1$.

18. With vertex $(-3, 2)$ and directrix $y + 1 = 0$.

19. With focus $(0, 6)$, axis Oy, and latus rectum 8. (Two answers.)

20. With focus $(2, 3)$, axis parallel to Ox, and latus rectum 12.

21. With vertex on Oy, axis parallel to Ox, and passing through $(-4, 1)$, $(-1, -1)$. *Ans.* $(y + 3)^2 = -4x$; $(y + \frac{1}{3})^2 = -\frac{4}{9}x$.

22. With vertex on Ox, axis parallel to Oy, and passing through $(2, 3)$, $(-1, 12)$. *Ans.* $(x - 5)^2 = 3y$; $(x - 1)^2 = \frac{1}{3}y$.

23. With vertex on the line $y = 2$, axis parallel to Oy, latus rectum 6, and passing through $(0, 8)$. *Ans.* $(x \pm 6)^2 = 6(y - 2)$.

24. With axis parallel to Ox, latus rectum 1, and passing through $(-6, 4)$, $(9, 1)$. *Ans.* $(y - 5)^2 = x + 7$; $y^2 = -(x - 10)$.

25. Show in detail how to construct points of a parabola by ruler and compass if the focus and directrix are given.

Solve the following problems geometrically (by ruler and compass).

26. Given a parabola with its vertex marked, construct the axis.

27. Given a parabola with its focus, construct the directrix.

28. Given the directrix and two points of a parabola, find the focus. How many solutions are there, in general? Discuss exceptional cases.

29. Given the focus and two points of a parabola, find the directrix.

30. Given the directrix, the tangent at the vertex (vertex not marked), and one point of a parabola, construct the focus.

31. Given a parabola with its vertex, find the focus.

43. Reduction to standard form. The equation

$$(1) \qquad Cy^2 + Dx + Ey + F = 0 \qquad (C \neq 0)$$

can be reduced to form (2) or (3), § 42, by dividing by C and completing the square in y; similarly the equation

$$(2) \qquad Ax^2 + Dx + Ey + F = 0 \qquad (A \neq 0)$$

can be reduced to form (4) or (5), § 42. Exception arises when, in (1), $D = 0$ or, in (2), $E = 0$, in which cases the equation represents two parallel or coincident lines,* or has no locus. In summary, we have the

THEOREM: *An equation of the second degree in which the xy-term is missing and only one square term is present represents a parabola with its axis parallel to a coördinate axis (exceptionally, two parallel or coincident lines, or no locus).*

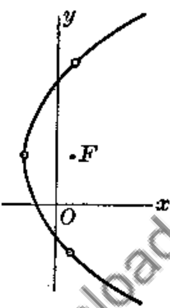

FIG. 46

Examples: (a) Reduce the equation

$$4y^2 - 24x - 12y = 15$$

to a standard form, and trace the curve.

First divide by 4:

$$y^2 - 6x - 3y = \tfrac{15}{4}.$$

Transpose the term in x and complete the square in y:

$$y^2 - 3y + \tfrac{9}{4} = 6x + \tfrac{15}{4} + \tfrac{9}{4},$$

or

$$(y - \tfrac{3}{2})^2 = 6(x + 1).$$

Hence the vertex is at $(-1, \frac{3}{2})$ and the curve opens to the right, with $a = \frac{3}{2}$.

* Geometrically, of course, two coincident lines cannot be distinguished from a single line. But for various reasons the terminology here adopted is preferable. First, we say in algebra that when $D^2 - 4AF = 0$, the quadratic equation $Ax^2 + Dx + F = 0$ has, not one root, but two equal roots; clearly the analogous statement is that the locus of this equation is not one line but two coincident lines. Second, if we say that the locus is a single line, then the converse theorem of § 26, as stated, is contradicted; and that theorem cannot be stated in any other form without obscuring the essential truth.

(*b*) Find the equation of a parabola with axis parallel to Ox and passing through (6, 1), (– 2, 3), (16, 6).

It will be most convenient to assume the equation in form (1), taking * $C = 1$:

$$y^2 + Dx + Ey + F = 0.$$

Substituting the coördinates of the given points in turn, we obtain the following equations to determine D, E, F:

$$1 + 6D + E + F = 0,$$
$$9 - 2D + 3E + F = 0,$$
$$36 + 16D + 6E + F = 0.$$

The solution may be completed by the student.

EXERCISES

In the following cases, reduce to a standard form, locate the vertex, the ends of the latus rectum, and a few other points, and trace the curve.

1. $y^2 + 8x + 8 = 0.$

2. $y^2 - 5x + 10 = 0.$

3. $x^2 - 4x - 4y = 0.$

4. $x^2 + 6x - y - 2 = 0.$

5. $y^2 + 5x - 3y = 0.$

6. $x^2 - 2x - y = 3.$

7. $2x^2 + 4x + y + 6 = 0.$

8. $3y^2 - 2x - 6y = 0.$

9. $3x^2 + 2x - y + 1 = 0.$

10. $5x^2 - 6x + 2y + 1 = 0.$

11. $x^2 - 20x + y = 20.$

12. $y^2 - 100x - 200y + 200 = 0.$

13. $12y^2 - 2x - y = 0.$

14. $8x^2 + x + 2y = 1.$

15. Find the equation of a parabola with axis parallel to Ox and passing through (1, – 1), (9, 1), (9, – 2). *Ans.* $4y^2 - x + 4y + 1 = 0.$

16. Find the equation of a parabola with axis parallel to Oy and passing through (1, 1), (3, 0), (4, – 4). *Ans.* $7x^2 - 25x + 6y + 12 = 0.$

17. Find the equation of a parabola with axis parallel to Oy and passing through (4, 5), (– 2, 11), (– 4, 21). *Ans.* $x^2 - 4x - 2y + 10 = 0.$

18. Find the locus of the center of a circle which is tangent to the line $x = 10$ and passes through (2, 1). *Ans.* $y^2 + 16x - 2y = 95.$

19. Find the equation of a circle through (0, 5), (3, 4), tangent to the line $y + 5 = 0$. *Ans.* $x^2 + y^2 = 25$; $(x - 60)^2 + (y - 180)^2 = (185)^2.$

* Cf. the first footnote, p. 52, and the example, § 36.

Trace the given curves, and find their points of intersection.

20. $x^2 + 6x - 3y + 11 = 0, \quad x^2 + y^2 + 2x - 2y + 1 = 0.$

21. $y^2 + x - 2y = 0, \quad x^2 = y.$

22. $x^2 - 4x - 4y = 4, \quad x^2 = 4x + 4y.$ *Ans.* No intersection.

23. $x^2 = 3y, \quad x^2 = y - 2.$

24. $x^2 + x = y, \quad y^2 + 12x = 8y.$ *Ans.* (0, 0), (1, 2) twice, (− 4, 12).

25. $x^2 - 2x + y + 2 = 0, \quad y^2 + 6x + 9y + 2 = 0.$
Ans. (1, − 1), (2, − 2), (3, − 5), (− 2, − 10).

26. $y^2 - x - 4y + 1 = 0, \quad x^2 + 6x - 8y + 25 = 0.$

27. $x^2 - 6x + 2y + 2 = 0, \quad y^2 - 2x - 4y + 7 = 0.$

28. Find the equation of a circle through the points of intersection of the parabolas of Ex. 24. (See § 37.) Verify by substitution that the circle actually passes through the points.

29. Solve Ex. 28 for the parabolas of Ex. 25.

30. If four points are common to two parabolas whose axes are at right angles, prove that the four points lie on a circle. (Use equations (1), (2) of § 43, and see § 37.)

44. Quadratic functions. A function (§ 28) of the form

$$y = ax^2 + bx + c \qquad (a \neq 0) \tag{1}$$

is called a *quadratic function*. Since this equation is of the form (2), § 43, it appears that *the graph of a quadratic function is always a parabola with vertical axis*, or (§ 28) a portion of such a parabola.

As x changes, many functions increase to a *maximum* value, decreasing thereafter, or decrease to a *minimum* and then begin to increase. In such cases, it is usually a problem of prime importance to determine the maximum or minimum value. To solve this problem for functions in general, the differential calculus is required; for the quadratic function it may evidently be solved by merely finding the vertex of the parabola (1).

Example: A ball is thrown upward with a velocity of 40 ft. per sec. (*a*) Find how far it will rise. (*b*) If it starts at a height of 24 ft., find when it will strike the ground.

When a body moves in a vertical line with no force acting except the earth's attraction, it is shown in mechanics that the distance x (in feet) from the starting point after time t (in seconds) is approximately

(2) $$x = v_0 t - 16t^2,$$

where v_0 is the initial velocity, both x and v_0 being *positive upward*. Take $v_0 = 40$, and standardize (2):

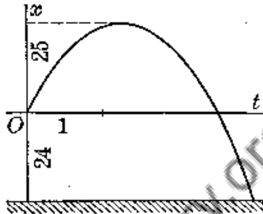

FIG. 47

$$(t - \tfrac{5}{4})^2 = -\tfrac{1}{16}(x - 25).$$

Thus the ball rises $\frac{5}{4}$ sec., to a height of 25 ft. above the starting point. Putting $x = -24$ in (2), we find

$$2t^2 - 5t - 3 = 0, \qquad (2t + 1)(t - 3) = 0,$$

whence the ball strikes the ground after 3 sec.

EXERCISES

In each case draw the curve, and indicate the portion that has a meaning.

1. A ball is thrown upward with a velocity of 64 ft. per sec. Find (*a*) how far and for how long a time it will rise; (*b*) when it will be halfway between the starting point and the highest point. *Ans.* (*b*) 0.6 sec., 3.4 sec.

2. A ball is thrown downward from a height of 10 ft., with a velocity of 12 ft. per sec. Find when it will strike the ground. *Ans.* 0.5 sec.

3. Express the total surface (including both ends) of a right circular cylinder of unit altitude, as a function of the radius.

4. Express the total surface of a right circular cone of slant height 2, as a function of the radius.

5. Find the radius of the circular cross-section cut from a sphere of radius r by a plane passing at a distance h from the center. Taking $r = 1$, express the area of the section as a function of h.

6. Express $\cos 2\theta$ as a function of $\cos \theta$.

7. When the load is uniformly distributed horizontally, a suspension-bridge cable hangs in a parabolic arc. If the bridge is 200 ft. long, the towers 40 ft. high, and the cable 15 ft. above the floor of the bridge at the midpoint, find the equation of the parabola with the midpoint of the bridge as origin; also the height 50 ft. from the middle.

8. A rectangular lot is to be inclosed by 100 yd. of fencing. Express the area as a function of one of the sides, and find the dimensions of the largest lot that can be inclosed. *Ans.* 25×25 yd.

9. A rectangular lot is to be fenced off along the bank of a river. If no fence is needed along the river, find the dimensions of the largest lot that can be inclosed with 100 yd. of fencing. *Ans.* 25×50 yd.

10. A rectangular field is to be inclosed, and divided into three lots by parallels to one of the sides. Find the dimensions of the largest field that can be inclosed with 1000 yd. of fencing. *Ans.* 250×125 yd.

11. A triangular lot has 60 ft. frontage on one street, 80 ft. on another street at right angles to the first. Find the dimensions of the largest rectangular building that can be erected facing one of the streets. (§ 30.)

12. A right circular cylinder is inscribed in a right circular cone of radius 4 in., altitude 12 in. Find the radius of the cylinder if its convex surface is a maximum. (§ 30.) *Ans.* 2 in.

13. Solve Ex. 12 if the total surface of the cylinder is a maximum.

14. A rectangular lot is to be fenced off along a highway. If the fence on the highway costs \$1.50 per yd., on the other sides \$1 per yd., find the size of the largest lot that can be inclosed for \$100. *Ans.* 20×25 yd.

15. When a projectile is thrown with an initial velocity v_0 inclined at an angle α to the horizontal, the equation of its path (with all resistances neglected) is

$$y = x \tan \alpha - \frac{16x^2}{v_0^2} \sec^2 \alpha.$$

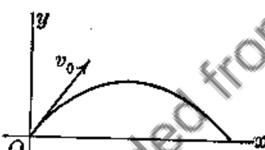

FIG. 48

If a ball thrown from the ground with $v_0 = 60$ ft. per sec. strikes the ground again 90 ft. away, find α, and plot the possible paths. Find the highest point for each value of α. If the ball is to be caught at a height of 5 ft., where should the catcher stand? *Ans.* $\alpha = 26.°6$ or $63.°4$.

16. A projectile (see Ex. 15) thrown with initial velocity $v_0 = 64$ ft. per sec. strikes a wall at distance $x = 32$ ft. away. Plot y as a function of $\tan \alpha$. What value of α gives the greatest height on the wall, and what maximum height is attainable? *Ans.* $\alpha = 76°$; 60 ft.

17. In the formula of Ex. 14, p. 42, with $W = 2$, $w = 1$, $l_2 = 4$, plot F as a function of l_1. What is the meaning of the l_1-intercept?

18. When a body is thrown vertically upward with velocity v_0, the velocity at any height x is given approximately by the formula

$$v^2 = v_0^2 - 64x.$$

Taking $v_0 = 64$, draw the graph of v as a function of x. Find the velocity at a height of 48 ft. Why are there two values of v?

II. The Ellipse

45. Ellipse referred to its axes: first standard form. By § 39, the ellipse is *the conic section for which* $e < 1$.

Let F be the focus and λ the directrix. The line FM through the focus perpendicular to the directrix intersects the curve in two points,* say V, V'. These points are the *vertices*, and C, the midpoint of VV', is the *center*. Let us set

$$CV = a,$$
$$CF = c,$$
$$CM = d.$$

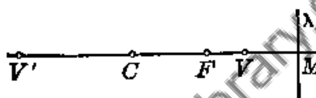

Fig. 49

Then, applying (1), § 39, to the points V, V', we have

$$a - c = e(d - a), \qquad a + c = e(d + a).$$

By subtraction and addition we find

$$2c = 2CF = 2ae, \qquad CF = ae,$$

$$2ed = 2e \cdot CM = 2a, \qquad CM = \frac{a}{e}.$$

Let us now (Fig. 50) take the center at (0, 0) and focus at F_1: $(ae, 0)$, so that the directrix is the line

$$x = \frac{a}{e}.$$

Then, if P: (x, y) is a point on the curve, we have (§ 39)

$$\sqrt{(x - ae)^2 + y^2} = e\left(\frac{a}{e} - x\right) = a - ex,$$

$$x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2,$$
$$x^2(1 - e^2) + y^2 = a^2(1 - e^2),$$

(1) $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

* For FM can be divided both internally and externally in the ratio e.

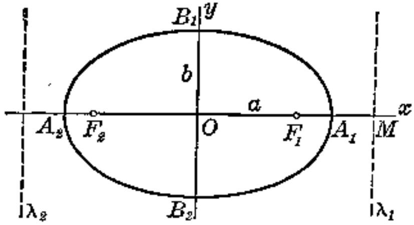

FIG. 50

For simplicity, let us put

(2) $\quad b^2 = a^2(1 - e^2),$

thus reducing equation (1) to the *standard form*

(3) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

From (3) we see that:

(*a*) The curve is symmetric with respect to Ox and Oy. From the latter statement it follows that there is *a second focus, the point* $(-ae, 0)$, and *a second directrix, the line* $x = -\dfrac{a}{e}$.

(*b*) The curve intersects the axes at $(\pm a, 0)$, $(0, \pm b)$.

(*c*) The equation when solved for y has the form

(4) $$y = \pm \frac{b}{a}\sqrt{a^2 - x^2},$$

which shows that y is imaginary when x is numerically greater than a. Similarly, x is imaginary when y is numerically greater than b.

It is convenient occasionally to speak of the two lines of symmetry as the *axes* of the curve, but usually we shall consider the axes to be the *segments* of these lines included within the curve: the segment A_2A_1 is the *major axis*, the segment B_2B_1 the *minor axis*.

It follows from (2) that, for the ellipse, *a is always greater than b* — hence the terms "major" and "minor."

From equation (3), the ends of the axes may be plotted immediately. By (2), we find

$$ae = \sqrt{a^2 - b^2},$$

so that *the distance from center to foci is* $\sqrt{a^2 - b^2}$. By (4), when

$$x = ae, \qquad y = \pm \frac{b}{a}\sqrt{a^2 - a^2e^2} = \pm \frac{b^2}{a},$$

so that *the latus rectum is* $\frac{2b^2}{a}$. The ends of the latera recta may then be plotted. This gives a total of eight points on the curve, from which a fairly accurate sketch can be made.

It is easily seen that the equation

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$

represents the *ellipse with center at the origin and major axis in Oy*.

40. Another definition of the ellipse. The following property of the ellipse is often used as a *definition:*

An ellipse is the locus of a point which moves so that the sum of its distances from two fixed points is constant. The fixed points are the *foci*; the constant sum is the *major axis*.

Let the foci be $F_1 : (ae, 0)$, $F_2 : (-ae, 0)$, the directrices $x = \pm \frac{a}{e}$, and $P : (x, y)$ any point of an ellipse. By (1), § 39,

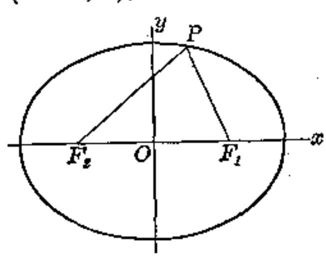

FIG. 51

$$F_1P = e\left(\frac{a}{e} - x\right) = a - ex,$$

$$F_2P = e\left(\frac{a}{e} + x\right) = a + ex,$$

$$\therefore \quad F_1P + F_2P = 2a.$$

To complete the proof of equivalence of the two definitions, it must be shown that, when a point moves so that the sum of its distances from two fixed points is constant, its locus is an ellipse as defined in § 39. This part of the proof will be left to the student (cf. Ex. 15, p. 28).

EXERCISES

In Exs. 1–6, find the center and foci, plot the ends of the axes and of the latera recta, and draw the curve. Find the eccentricity and the directrices.

1. $\frac{x^2}{169} + \frac{y^2}{144} = 1.$ **2.** $\frac{x^2}{100} + \frac{y^2}{36} = 1.$

3. $\frac{x^2}{2} + \frac{y^2}{3} = 1.$ **4.** $\frac{x^2}{9} + \frac{y^2}{12} = 1.$

5. $3x^2 + 5y^2 = 15.$ **6.** $4x^2 + 2y^2 = 9.$

Find the equations of the following ellipses, assuming form (3), § 45.

7. Major axis 9, distance between foci 8.

8. Latus rectum 4, distance between foci $4\sqrt{2}$. *Ans.* $x^2 + 2y^2 = 16.$

9. Eccentricity $\frac{1}{2}$, distance between foci 1. *Ans.* $3x^2 + 4y^2 = 3.$

10. Distance between foci $8\sqrt{6}$, rectangle on the axes of area 80.

11. Eccentricity $\frac{2}{3}$, latus rectum $\frac{2}{3}$. *Ans.* $25x^2 + 45y^2 = 9.$

12. Distance between foci 2, between directrices 8.

13. Passing through (4, 3), (6, 2). *Ans.* $x^2 + 4y^2 = 52.$

14. Passing through (1, 2), (3, 1). *Ans.* $3x^2 + 8y^2 = 35.$

15. Passing through (2, 3), latus rectum three times the distance from center to focus. *Ans.* $3x^2 + 4y^2 = 48.$

16. Distance between the directrices $\frac{2}{3}\sqrt{21}$, the rectangle on the axes of area $\frac{8}{7}\sqrt{7}$. *Ans.* $4x^2 + 7y^2 = 4$; $x^2 + 7y^2 = 2.$

17. Latus rectum $\frac{60}{19}$, distance between directrices $2\sqrt{19}$.
Ans. $15x^2 + 19y^2 = 60$; $10x^2 + 19y^2 = 90.$

18. Distance between foci $\frac{4}{3}\sqrt{33}$, passing through (2, 1).
Ans. $x^2 + 12y^2 = 16.$

19. Prove that as e approaches 0, the ellipse approaches the circle as a limiting form. (Let e approach 0 in (1), § 45. As this happens, how do the focus and directrix move?)

20. A line segment of fixed length moves with its ends following two perpendicular lines. The line is divided by a point P into two segments of lengths a, b. Find the locus of P. *Ans.* An ellipse.

21. A point moves so that the sum of its distances from (6, 0), (− 6, 0) is 16. Find the equation of its locus in two ways (§§ 20, 46). Draw the curve.

22. A point moves so that the sum of its distances from (0, 2), (0, − 2) is 6. Find the equation of its locus in two ways, and draw the curve.

23. A circle is tangent to the circle $(x + 1)^2 + y^2 = 9$, and passes through (1, 0). Find the locus of its center. *Ans.* $20x^2 + 36y^2 = 45.$

24. A moving circle is tangent to the fixed circles $(x - c)^2 + y^2 = c^2$, $(x + c)^2 + y^2 = 16c^2$. Find the locus of its center.

Ans. $84x^2 + 100y^2 = 525c^2$; $20x^2 + 36y^2 = 45c^2$.

25. A moving circle is tangent to the fixed circles $x^2 + (y - 2c)^2 = c^2$, $x^2 + (y + 2c)^2 = 25c^2$. Find the locus of its center. Draw the figure.

Solve the following problems by ruler and compass.

26. Given an ellipse with its center, construct the axes.

27. Given an ellipse with its axes, construct the foci.

28. Given an ellipse with its axes, construct the directrices.

29. Given the foci and major axis of an ellipse, show how to construct points of the curve by ruler and compass. (§ 46.)

30. Given the foci and one point of an ellipse, construct the axes.

31. Given the foci and minor axis of an ellipse, construct the major axis.

32. Given one focus, the direction of the major axis, and the lengths of both axes, construct the other focus.

47. Other standard forms. By a method similar to that of § 42, we may establish the following results:

The equation of an ellipse with center at (h, k) *is, if the major axis is parallel to Ox,*

$$(1) \qquad \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1; \qquad (a > b)$$

if the major axis is parallel to Oy,

$$(2) \qquad \frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1. \qquad (a > b)$$

48. Reduction to standard form. Given the equation

$$(1) \qquad Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where A and C have *the same sign*, it is easily seen that, by completing the squares in x and y, the equation can in general be reduced to one of the standard forms of § 47. There are two exceptional cases: When the left member is written as the sum of two squares, the right member may be 0, or it may be negative. In the former case the locus is evidently the single point (h, k), — the so-called "point-ellipse"; in the latter case there is no locus. Hence the

THEOREM: *An equation of the second degree in which the xy-term is missing and the coefficients of x^2 and y^2 have the same sign represents an ellipse with axes parallel to the coördinate axes (exceptionally, a single point, or no locus).*

Example: Trace the curve

$$9x^2 + 4y^2 - 36x + 8y + 31 = 0.$$

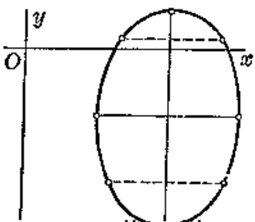

FIG. 52

Transpose the constant and complete the squares:

$$9(x^2 - 4x + 4) + 4(y^2 + 2y + 1) = -31 + 36 + 4,$$

$$9(x - 2)^2 + 4(y + 1)^2 = 9,$$

$$\frac{(x-2)^2}{1} + \frac{(y+1)^2}{\frac{9}{4}} = 1.$$

The center is at $(2, -1)$, major axis parallel to Oy, semi-axes $a = \frac{3}{2}$, $b = 1$, distance from center to foci $\sqrt{a^2 - b^2} = \sqrt{\frac{5}{4}} = \frac{1}{2}\sqrt{5}$, latus rectum $\frac{2b^2}{a} = \frac{4}{3}$.

EXERCISES

In each of the following cases, find the center and foci, plot the ends of the axes and of the latera recta, and draw the curve.

1. $\frac{(x-3)^2}{4} + \frac{(y+1)^2}{3} = 1.$
2. $\frac{(x+\frac{3}{2})^2}{\frac{9}{4}} + \frac{(y-1)^2}{\frac{1}{4}} = 1.$
3. $\frac{x^2}{\frac{3}{4}} + \frac{(y-\frac{1}{2})^2}{1} = 1.$
4. $\frac{(x-6)^2}{8} + \frac{(y-4)^2}{12} = 1.$
5. $x^2 + 3y^2 + 6x + 6 = 0.$
6. $x^2 + 2y^2 - 2x - 4y = 1.$
7. $8x^2 + y^2 - 16x + 2y = 0.$
8. $2x^2 + y^2 + 8x + 4y = 0.$
9. $4x^2 + y^2 = 4cx.$
10. $x^2 + 5y^2 = 10cy.$
11. $4x^2 + 5y^2 + 16x - 20y + 31 = 0.$

   Ans. $\frac{(x+2)^2}{\frac{5}{4}} + \frac{(y-2)^2}{1} = 1.$
12. $7x^2 + 8y^2 - 28x + 80y + 172 = 0.$
13. $7x^2 + 2y^2 - 28x + 4y + 16 = 0.$

   Ans. $\frac{(x-2)^2}{2} + \frac{(y+1)^2}{7} = 1.$
14. $3x^2 + 7y^2 - 12x + 28y + 19 = 0.$

III. The Hyperbola

49. Hyperbola referred to its axes: first standard form. By § 39, the hyperbola is *the conic for which* $e > 1$.

The derivation of (1), § 45, does not depend upon the fact that $e < 1$; thus we know at once that, with the axes chosen as in § 45, the equation of the hyperbola is

(1) $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

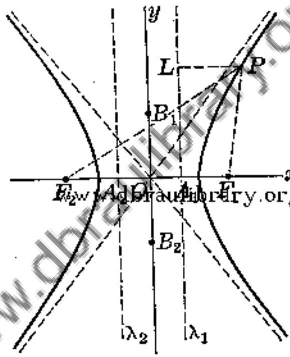

FIG. 53

But, since now $e > 1$, the constant $a^2(1 - e^2)$ is negative; therefore to make b real, we set

(2) $$b^2 = a^2(e^2 - 1),$$

and write (1) in the form

(3) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This equation shows that:

(*a*) The curve is symmetric with respect to both axes. Hence there are two foci, the points $(\pm ae, 0)$, and two directrices, the lines $x = \pm \frac{a}{e}$.

(*b*) The curve crosses Ox at $(\pm a, 0)$; the intersections with Oy are imaginary.

(*c*) The equation when solved for y has the form

(4) $$y = \pm \frac{b}{a}\sqrt{x^2 - a^2},$$

which shows that y is imaginary if and only if $x^2 < a^2$, i.e. if $-a < x < a$. The curve therefore consists of two disconnected branches, one lying to the right of the line $x = a$, the other to the left of the line $x = -a$.

(*d*) *The latus rectum is* $\frac{2b^2}{a}$.

As in the case of the ellipse, the lines of symmetry are sometimes spoken of as the *axes* of the curve, but unless the contrary is indicated the axes will be considered to be the *segments* A_2A_1 of length $2a$ and B_2B_1 of length $2b$: the former is the *transverse axis*, the latter the *conjugate axis*. The conjugate axis does not intersect the curve, but plays an important part in its theory. The ends A_2, A_1 of the transverse axis are the *vertices*; the point of intersection of the axes is the *center*.

It appears from (2) that the *distance from the center to the foci* is

(5) $$ae = \sqrt{a^2 + b^2}.$$

It should also be noted that in the case of the hyperbola b may be *greater than, equal to*, or *less than* a, according to the value of e.

It is easily seen that the equation

(6) $$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

represents a *hyperbola with transverse axis in Oy.*

50. Asymptotes. When the standard hyperbola

(1) $$b^2x^2 - a^2y^2 = a^2b^2$$

and the straight line

(2) $$bx - ay = 0$$

are drawn on the same axes, the figure indicates that the hyperbola approaches the straight line more and more closely as the distance from the center increases, but without ever reaching the line. To prove this, let $P : (x_1, y_1)$ be a point on the hyperbola in the first or third quadrant. The distance from P to the line (2) is (§ 32)

(3) $$d = \frac{bx_1 - ay_1}{\sqrt{a^2 + b^2}}.$$

Since P is on the curve, its coördinates satisfy (1):

$$b^2x_1^2 - a^2y_1^2 = a^2b^2,$$

or

$$(bx_1 - ay_1)(bx_1 + ay_1) = a^2b^2,$$

whence

$$bx_1 - ay_1 = \frac{a^2b^2}{bx_1 + ay_1}.$$

Substituting this value of $bx_1 - ay_1$ in (3), we find

$$d = \frac{a^2b^2}{\sqrt{a^2+b^2}} \cdot \frac{1}{bx_1 + ay_1}.$$

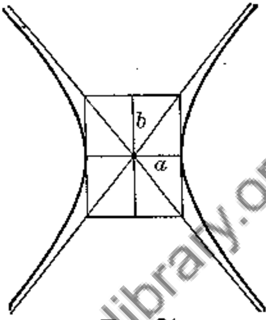

FIG. 54

Evidently d can never equal 0, but as P recedes, so that x_1, y_1 both increase numerically without limit, d becomes smaller and smaller, approaching the limit 0.

A similar result is easily established for the line

(4) $$bx + ay = 0$$

when P lies in the second or fourth quadrant.

The lines (2) and (4), or as usually written,

(5) $$y = \pm \frac{b}{a}x,$$

are called the *asymptotes** of the hyperbola. They are of great importance both in tracing the curve and in studying its properties.

From (5) we derive the following result, which gives a convenient method for drawing the asymptotes of any hyperbola whose axes are given:

The asymptotes of a hyperbola are the diagonal lines of the rectangle whose center is the center of the curve and whose sides are parallel and equal to the axes of the curve.

* A general definition of this term will be given in § 64.

To trace a hyperbola whose equation is in a standard form, we *plot the vertices and the ends of the latera recta,* and *draw the asymptotes.* These data suffice for a fairly satisfactory sketch; if greater accuracy is required, more points must be plotted.

51. Equilateral, or rectangular, hyperbola. The hyperbola for which a and b are equal is called, on account of the equality of the semi-axes, the *equilateral* hyperbola. Since the rectangle of Fig. 54 is in this case a square, the asymptotes of the equilateral hyperbola are at right angles: for this reason it is also called the *rectangular* hyperbola. Since $ae = \sqrt{a^2 + b^2}$, the eccentricity of the equilateral hyperbola is $e = \sqrt{2}$.

When the hyperbola is equilateral, equations (3) and (6) of § 49 evidently assume the respective forms

$$x^2 - y^2 = a^2,$$
$$y^2 - x^2 = a^2.$$

52. Another definition of the hyperbola. A property of the hyperbola frequently used as a *definition* of the curve is as follows (cf. § 46):

A hyperbola is the locus of a point which moves so that the difference of its distances from two fixed points is constant. The fixed points are the *foci* and the constant difference is the *transverse axis.*

That is, in Fig. 53, p. 79,

(1) $$F_2P - F_1P = 2a.$$

The proof may be carried out by the student.

EXERCISES

Locate the center, vertices, foci, and ends of the latera recta, draw the asymptotes, and trace the curve. Determine the eccentricity and write the equations of the directrices and asymptotes.

1. $\frac{x^2}{4} - \frac{y^2}{2} = 1.$ 2. $\frac{x^2}{4} - \frac{y^2}{5} = 1.$

3. $\frac{y^2}{9} - \frac{x^2}{16} = 1.$ 4. $\frac{y^2}{6} - \frac{x^2}{3} = 1.$

5. $\frac{x^2}{16} - \frac{y^2}{1} = 1.$ 6. $\frac{y^2}{25} - \frac{x^2}{4} = 1.$

Find the equations of the following hyperbolas, assuming the center at O and transverse axis along Ox.

7. Latus rectum 36, distance between foci 24. *Ans.* $3x^2 - y^2 = 108.$

8. Eccentricity $2\sqrt{2}$ latus rectum 6. *Ans.* $49x^2 - 7y^2 = 9.$

9. Distance between foci 6, distance between directrices 4.

10. Passing through (2, 1), (4, 3). *Ans.* $2x^2 - 3y^2 = 5.$

11. Latus rectum $\frac{4}{3}$, slope of asymptotes $\pm \frac{1}{3}$. *Ans.* $x^2 - 9y^2 = 36.$

12. Foci $(\pm 4, 0)$, slope of asymptotes ± 3. *Ans.* $45x^2 - 5y^2 = 72.$

13. Latus rectum 18, distance between directrices 3.
Ans. $3x^2 - y^2 = 27.$

14. Latus rectum $\frac{9}{2}$, distance between directrices $\frac{32}{5}$.
Ans. $9x^2 - 16y^2 = 144.$

15. Distance between directrices 1, the rectangle on the axes of area 2.
Ans. $2x^2 - 2y^2 = 1.$

16. Draw the graph of $\sec\theta$ as a function of $\tan\theta$.

17. Given a hyperbola with its axes, draw the asymptotes and find the foci geometrically.

18. Given a hyperbola with its axes, find the directrices.

19. Prove analytically that a line parallel to an asymptote of a hyperbola intersects the curve in one and only one point.

20. Prove analytically that the product of the distances of any point of a hyperbola from its asymptotes is constant.

21. A point moves so that the product of its distances from two intersecting lines is constant. Prove that its locus is a hyperbola having the given lines as asymptotes. (Take the lines $y = \pm mx$. Is the proof general?)

22. A point moves so that the difference of its distances from (3, 0), (− 3, 0) is 2. Find the equation of its locus in two ways. (§§ 20, 52.)

23. A point moves so that the difference of its distances from (0, 4), (0, − 4) is 6. Find the equation of its locus in two ways.

24. A circle is tangent to the circle $x^2 + y^2 + 2cx = 0$ and passes through $(c, 0)$. Find the locus of its center. *Ans.* $12x^2 - 4y^2 = 3c^2.$

25. A circle passes through a given point and is tangent to a given circle. Find the locus of its center, for all possible cases. (§§ 46, 52.)

26. A moving circle is tangent externally to the two fixed circles $(x - 2c)^2 + y^2 = c^2$, $(x + 2c)^2 + y^2 = 4c^2$. Find the locus of its center.

Ans. $60x^2 - 4y^2 = 15c^2$.

27. Given the foci and transverse axis of a hyperbola, show how to construct points of the curve by ruler and compass. (§ 52.)

28. Given the foci and one point of a hyperbola, construct the axes.

29. Given the foci and asymptotes of a hyperbola, find the vertices.

30. The sound of a gun and the ring of the ball on the target are heard simultaneously at a point P. What is the locus of P?

31. Prove equation (1), § 52.

53. Other standard forms. Formulas for the hyperbola analogous to those of § 47 for the ellipse are as follows:

The equation of a hyperbola with center at (h, k) is, if the transverse axis is parallel to Ox,

(1) $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1;$$

if the transverse axis is parallel to Oy,

(2) $$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

The proof is left to the student.

54. Reduction to standard form. The equation

(1) $$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where A and C have *opposite* signs, can evidently be reduced, by completing the squares in x and y, to one of the forms (1), (2) of § 53. The only exceptional case is the one in which, when the left member has been expressed as the difference of two squares, the right member reduces to 0:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 0.$$

This equation can be factored, and therefore represents two straight lines, intersecting at (h, k). Hence the

THEOREM: *An equation of the second degree in which the xy-term is missing and the coefficients of x^2 and y^2 have unlike signs represents a hyperbola with its axes parallel to the coordinate axes (exceptionally, two intersecting lines).*

Example: Trace the curve
$$x^2 - 2y^2 + 4x + 4y + 4 = 0.$$
Completing the squares in x and y, we get
$$(x + 2)^2 - 2(y - 1)^2 = -2,$$
or, dividing by -2,
$$\frac{(y - 1)^2}{1} - \frac{(x + 2)^2}{2} = 1.$$

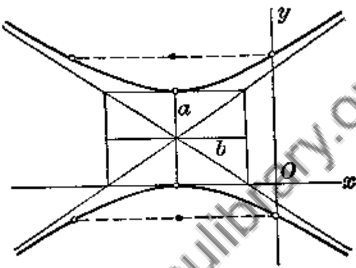

FIG. 55

This is a hyperbola with center at $(-2, 1)$ and transverse axis parallel to Oy; the semi-axes are $a = 1$, $b = \sqrt{2}$. The vertices are at the distance 1, the foci at the distance $\sqrt{a^2 + b^2} = \sqrt{3}$, above and below the center. The latus rectum is $\frac{2b^2}{a} = 4$, so that the ends of the latera recta are at the distance 2 to right and left of the foci. The asymptotes are the diagonals of the rectangle of sides $2a$, $2b$.

EXERCISES

Locate the center, vertices, foci, and ends of the latera recta, draw the asymptotes, and trace the curve.

1. $\frac{(x + 3)^2}{4} - \frac{(y - 1)^2}{8} = 1.$
2. $\frac{(x - 1)^2}{3} - \frac{y^2}{1} = 1.$
3. $\frac{(y + \frac{3}{2})^2}{1} - \frac{(x + \frac{1}{2})^2}{\frac{1}{4}} = 1.$
4. $\frac{(y - 6)^2}{36} - \frac{(x - 5)^2}{4} = 1.$
5. $5x^2 - 4y^2 = 20x + 24y + 36.$ *Ans.* $\frac{(x - 2)^2}{4} - \frac{(y + 3)^2}{5} = 1.$
6. $9x^2 - 16y^2 = 36x - 96y - 36.$ *Ans.* $\frac{(y - 3)^2}{9} - \frac{(x - 2)^2}{16} = 1.$
7. $x^2 - y^2 + 6x + 2y + 10 = 0.$ *Ans.* $\frac{(y - 1)^2}{2} - \frac{(x + 3)^2}{2} = 1.$

8. $2x^2 - 2y^2 + x - y - 1 = 0.$ *Ans.* $\dfrac{(x + \frac{1}{4})^2}{\frac{1}{2}} - \dfrac{(y + \frac{1}{4})^2}{\frac{1}{2}} = 1.$

9. $x^2 - 2y^2 + ay = 0.$

10. $4x^2 - y^2 - 2ax = 0.$

11. $4y^2 = x^2 - 3x + 4.$

12. $x^2 - 4y^2 + 4ax - 8ay = 0.$

13. $3x^2 - 4y^2 + 6x + 6y = 0.$

14. $4x^2 - 4y^2 + 4x - 8y = 1.$

15. $x^2 - 4y^2 + 2x + 8y = 3.$

16. $16x^2 - y^2 - 4x + y - 6 = 0.$

17. Show that if $C = -A$, equation (1) of § 54 represents an equilateral hyperbola (§ 51).

Trace the following curves, and find their points of intersection.

18. $x^2 - 9y^2 - 2x - 18y - 17 = 0, \quad x + 3y + 2 = 0.$

19. $4x^2 - y^2 - 8x - 2y - 1 = 0, \quad 2x - y = 3.$

20. $x^2 - y^2 + 3x - y + 8 = 0, \quad x^2 - y^2 + 4x - 4y + 16 = 0.$

Ans. $(-2, 2), (1, 3).$

21. $x^2 - y^2 + 5x + 3y + 2 = 0, \quad x^2 - y^2 + 2x = 4.$ *Ans.* $(-4, 2).$

22. $x^2 - y^2 + 2x + 4y = 7, \quad x^2 - y^2 + 4x + 4y = 1.$

Ans. $(-3, 2)$ twice.

CHAPTER VII

COÖRDINATE TRANSFORMATIONS

55. Translation of axes. We have had many illustrations of the fact that a problem may be greatly simplified by taking the axes in a convenient position. It may happen, however, that the position of the axes is predetermined by the statement of the problem; it is then desirable that we have methods for shifting the axes to some more suitable position. Such methods will now be developed.

Consider first the *translation* of axes, in which the axes are moved parallel to their original positions. Let Ox, Oy be the original axes, $O'x'$, $O'y'$ the new, and suppose the *new origin* O' to be *the point* (h, k) *referred to the old axes.* If we denote by (x, y) the coördinates of any point P in the original system, by (x', y') the coördinates of the same point in the new system, then it appears from the figure that the two sets of coördinates are connected by the formulas

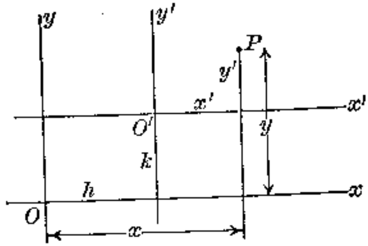

FIG. 56

$$(1) \qquad \begin{cases} x = x' + h, \\ y = y' + k. \end{cases}$$

An important problem is, being given the equation of a curve referred to any set of axes, to derive its equation referred to parallel axes (i.e. axes parallel to the original ones) through a point (h, k) as new origin. To do this, we have merely to *replace* x *and* y *in the equation of the curve by their values as given by* (1).

Example: Obtain the equation of the circle

$$x^2 + y^2 - 4x + 6y = 0$$

referred to parallel axes through $(2, -3)$ as new origin.

Substitute $x = x' + 2, \quad y = y' - 3$:

$$x'^2 + 4x' + 4 + y'^2 - 6y' + 9 - 4x' - 8 + 6y' - 18 = 0,$$

which reduces to $\quad x'^2 + y'^2 = 13.$

56. Rotation of axes. Let Ox, Oy and Ox', Oy' be two pairs of Cartesian axes with the same origin, and denote by ϕ the angle *through which the first pair must be rotated to come to coincidence with the second*, the angle ϕ being considered positive when measured *counterclockwise.* Let P have the coördinates x, y in the first system and x', y' in the second, so that

$$OM = x, \qquad MP = y,$$
$$OM' = x', \qquad M'P = y'.$$

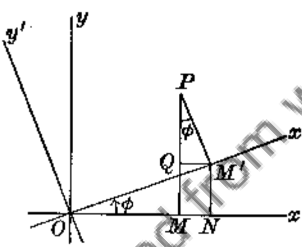

FIG. 57

Now

$$OM = ON - MN$$
$$= ON - QM';$$
$$MP = MQ + QP$$
$$= NM' + QP.$$

We thus have the formulas

(1) $$\begin{cases} x = x' \cos\phi - y' \sin\phi, \\ y = x' \sin\phi + y' \cos\phi. \end{cases}$$

Both in translation and in rotation of axes the original coördinates x, y are replaced by expressions of the *first degree* in x', y': from this it follows that *the degree of an equation is unchanged by translation or rotation of axes.*

Example: By rotating the axes through an angle ϕ such that $\tan\phi = -\frac{3}{4}$, prove that the equation

(2) $$(3x + 4y)^2 + 5x = 0$$

represents a parabola.

We see from the figure that $\cos\phi = -\frac{4}{5}$, $\sin\phi = \frac{3}{5}$. Substituting in (1), we get

$$x = -\tfrac{4}{5}x' - \tfrac{3}{5}y', \qquad y = \tfrac{3}{5}x' - \tfrac{4}{5}y'.$$

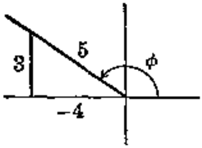

FIG. 58

These values substituted for x and y in (2) give

$$(-\tfrac{12}{5}x' - \tfrac{9}{5}y' + \tfrac{12}{5}x' - \tfrac{16}{5}y')^2 - 4x' - 3y' = 0,$$

or

$$25y'^2 - 4x' - 3y' = 0,$$

which by (1), § 43, is the equation of a parabola.

57. Combined translation and rotation. The formulas of §§ 55–56 taken together enable us to change the axes *from any position Ox, Oy to any other position O′x′, O′y′*. For we can first translate the axes to a parallel position with O' as new origin, and then rotate them through the angle necessary to bring them to the desired position.

An important consequence of the above remark is the following:

In investigating the purely geometric properties of any plane figure, we may without loss of generality place the figure in any convenient position with reference to the coördinate axes.*†

Thus, for instance, to prove the theorem of Ex. 20, p. 83, for *all* hyperbolas, we may use the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (not, however, $x^2 - y^2 = a^2$).

* By this is meant, of course, those properties which are intrinsic in the figure, and are therefore unaffected by its position with reference to the axes — e.g. the property of any triangle that its medians intersect in a trisection point.

† In § 11 this principle was assumed as intuitively evident, without formal proof, and has since been tacitly employed from time to time — e.g. in Exs. 19–20, p. 83.

EXERCISES

1. Find the coördinates of the points (3, 4), (5, − 3), (0, 2) referred to parallel axes through (3, − 1). Check by plotting.

2. Find the coördinates of the points (1, − 3), (6, 2), (0, − 5) referred to parallel axes through (5, − 3). Check by plotting.

3. In Exs. 8, 10, 12, p. 67, obtain the equation of the parabola referred to parallel axes through its vertex.

4. In Exs. 2, 4, 12, p. 78, obtain the equation of the ellipse referred to parallel axes through its center.

5. In Exs. 2, 4, 6, p. 85, obtain the equation of the hyperbola referred to parallel axes through its center.

6. Refer the curve $y = x^3 - 6x^2 + 12x - 5$ to parallel axes through (2, 3). Plot the curve on the new axes.

7. By translation of axes, remove the terms of first degree from the equation $x^2 - 4y^2 + 2x + 8y = 0$. (Substitute $x = x' + h$, $y = y' + k$, then equate to 0 the coefficients of the first-degree terms in x' and y'.)

Ans. $4y'^2 - x'^2 = 3$.

8. Solve Ex. 7 for the hyperbola $5x^2 - 4y^2 - 20x - 24y - 36 = 0$.

Ans. $5x'^2 - 4y'^2 = 20$.

9. By rotating the axes through 45°, prove that:

The equation $2xy = a^2$ represents an equilateral hyperbola asymptotic to the coördinate axes.

10. By rotating the axes through 45°, prove that the curve $x^2 + xy + y^2 = 1$ is an ellipse.

11. By rotating the axes through an angle ϕ such that $\tan \phi = 2$, prove that the curve $4x^2 - 4xy + y^2 + x + 2y = 0$ is a parabola.

12. By rotating the axes through an angle ϕ such that $\tan \phi = 3$, prove that the curve $4x^2 - 3xy = y$ is a hyperbola.

CHAPTER VIII

THE GENERAL EQUATION OF SECOND DEGREE

58. Removal of the product term. The most general equation of the second degree has the form

$$(1) \qquad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The results of §§ 43, 48, 54 show that when $B = 0$ this equation (if it has any locus at all) always represents a conic section, and it is reasonable to expect that the same will be true in case $B \neq 0$. This will be an established fact if we can show that by a translation or rotation of axes (§§ 55, 56) the term in xy can always be removed from equation (1). The transformation to be used is suggested by the following line of thought: If a conic lies with its axes inclined to the coördinate axes, as in Fig. 59, we can always rotate the axes through an angle ϕ (§ 56) to the position Ox', Oy' parallel to the axes of the curve, and we know that when this has been done the equation of the conic must be free of the product term. Hence, substitute for x and y in (1) the values given by (1), § 56:

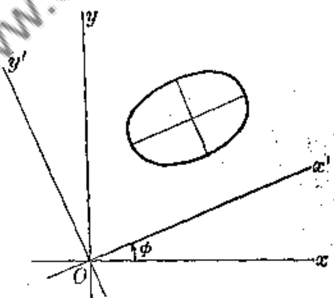

FIG. 59

$$(2) \qquad \begin{aligned} &A(x' \cos\phi - y' \sin\phi)^2 \\ &+ B(x' \cos\phi - y' \sin\phi)(x' \sin\phi + y' \cos\phi) \\ &+ C(x' \sin\phi + y' \cos\phi)^2 + D(x' \cos\phi - y' \sin\phi) \\ &\qquad + E(x' \sin\phi + y' \cos\phi) + F = 0. \end{aligned}$$

This equation will not contain the term in $x'y'$ if we equate the coefficient of that term to 0 and determine ϕ from the resulting formula. Upon expanding and collecting terms, it appears that this gives

$$2A \sin\phi \cos\phi - 2C \sin\phi \cos\phi - B\cos^2\phi + B\sin^2\phi = 0,$$

which may be written

(3) $$(A - C)\, 2 \sin\phi \cos\phi = B(\cos^2\phi - \sin^2\phi).$$

By the double-angle formulas of trigonometry, (3) becomes

(4) $$(A - C) \sin 2\phi = B \cos 2\phi,$$

or *

(5) $$\tan 2\phi = \frac{B}{A - C}.$$

Now for every possible value of $\tan 2\phi$, from $-\infty$ to $+\infty$, there is a value of 2ϕ between 0 and π, hence a value of ϕ between 0 and $\frac{1}{2}\pi$, so that it is always possible to determine a positive acute angle ϕ satisfying (4). If the curve (1) be referred to axes making this angle with the original ones, the resulting equation will be free of the product term, and its locus must be a conic. Hence the

THEOREM: *Every equation of the second degree (if it has a locus) represents a conic section, whose axes are inclined to the coördinate axes at the positive acute angle* † ϕ *given by formula* (5), *or, if* $A = C$, $B \neq 0$, *at an angle of* 45°.

Thus to trace the locus of any equation of the form (1), the first step is to determine $\tan 2\phi$ by (5), next to find $\cos 2\phi$ from a triangle (cf. the example, § 56), and then to obtain $\cos\phi$ and $\sin\phi$ by the formulas

* Formula (5) fails if $A = C$. But if $A = C$, $B \neq 0$, then $\phi = 45°$; for (4) gives $\cos 2\phi = 0$, $2\phi = 90°$. If $A = C$, $B = 0$, then (4) is true for all values of ϕ, which is to be expected. Why?

† There are of course infinitely many other values of ϕ, differing from this one by multiples of $\frac{1}{2}\pi$, but for definiteness we will agree to choose always the positive acute angle.

(6) $\cos\phi = \sqrt{\dfrac{1+\cos 2\phi}{2}}, \qquad \sin\phi = \sqrt{\dfrac{1-\cos 2\phi}{2}}.$

The values of $\cos\phi$ and $\sin\phi$, when substituted in (1), § 56, give us the expressions for x and y in terms of the new coördinates. Substituting these expressions in the original equation, we have the equation of the curve referred to the new axes. This equation will contain no xy-term, so that we may employ the methods of §§ 43, 48, 54.

Example: Trace the curve

(7) $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0.$

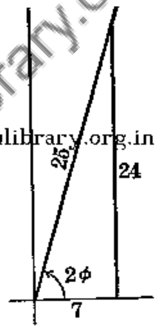

FIG. 60

Here we have

$$\tan 2\phi = \frac{-24}{9-16} = \frac{24}{7},$$

whence

$$\cos 2\phi = \tfrac{7}{25},$$

and by (6),

$$\cos\phi = \tfrac{4}{5}, \qquad \sin\phi = \tfrac{3}{5}.$$

Thus by § 56,

$$x = \tfrac{1}{5}(4x' - 3y'), \qquad y = \tfrac{1}{5}(3x' + 4y').$$

In terms of the new coördinates equation (7) becomes

$$25y'^2 - 75x' - 70y' + 19 = 0,$$

or, in the standard form,

$$(y' - \tfrac{7}{5})^2 = 3(x' + \tfrac{2}{5}).$$

This is a parabola with axis parallel to Ox', opening in the positive direction, and with its vertex at the point $(-\frac{2}{5}, \frac{7}{5})$ referred to the new axes. The x'-axis has a slope $\tan\phi = \frac{3}{4}$; after drawing the new axes the curve is traced as in Fig. 61.

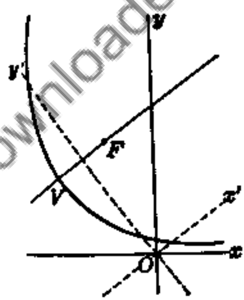

FIG. 61

EXERCISES

Remove the xy-term by rotation of axes, reduce the resulting equation to a standard form, and trace the curve on the new axes.

1. $xy = 2.$ 2. $xy + 8 = 0.$

3. $5x^2 - 6xy + 5y^2 = 8.$ *Ans.* $x'^2 + 4y'^2 =$ [illegible]

4. $2x^2 + 2xy + 2y^2 = 3.$ *Ans.* $3x'^2 + y'^2 =$ [illegible]

5. $x^2 + 6xy + y^2 + 8 = 0.$ *Ans.* $y'^2 - 2x'^2 =$ [illegible]

6. $3x^2 - 10xy + 3y^2 + 32 = 0.$ *Ans.* $x'^2 - 4y'^2 =$ [illegible]

7. $7x^2 + 12xy - 2y^2 = 10.$ *Ans.* $2x'^2 - y'^2 =$ [illegible]

8. $41x^2 - 84xy + 76y^2 = 208.$ *Ans.* $x'^2 + 8y'^2 =$ [illegible]

9. $19x^2 + 6xy + 11y^2 = 20.$ *Ans.* $2x'^2 + y'^2 =$ [illegible]

10. $4xy - 3y^2 = 8.$ *Ans.* $x'^2 - 4y'^2 =$ [illegible]

11. $x^2 + 3xy + 5y^2 = 22.$ *Ans.* $11x'^2 + y'^2 =$ [illegible]

12. $7x^2 - 12xy + 2y^2 = 44.$ *Ans.* $11y'^2 - 2x'^2 =$ [illegible]

13. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0.$ *Ans.* $x'^2 - 2x' -$ [illegible]

14. $11x^2 - 24xy + 4y^2 + 6x + 8y = 10.$ *Ans.* $4y'^2 - (x' - 1)^2 =$ [illegible]

15. Represent $\tan(\theta + 45°)$ as a function of $\tan\theta$. (Expand by the addition formula; put $\tan\theta = x$, $\tan(\theta + 45°) = y$.)

16. Draw a curve from which $\sin(\theta + 60°)$ may be read off if $\sin\theta$ is given. (Note the suggestion in Ex. 15.)

17. For the lever of Ex. 13, p. 42, if a weight of 5 lbs. is placed 5 ft. from the fulcrum, and if the lever weighs $1\frac{1}{2}$ lbs. per ft., draw the graph of F as a function of l. Estimate roughly the most advantageous length of lever to use.

18. For the lever of Ex. 14, p. 42, if a weight of 6 lbs. is 4 ft. from the fulcrum, and if the lever weighs 12 oz. per ft., graph F as a function of l. What is the best length of lever to use? Does the graph have a meaning in the fourth quadrant?

19. Prove that if a line is parallel to the axis of a parabola, it intersects the curve in one and only one point.

20. Prove that the equation $x^{1/2} \pm y^{1/2} = \pm a^{1/2}$ represents a parabola, and trace the curve.

CHAPTER IX

TANGENTS AND NORMALS

90. Tangents to plane curves. A straight line that intersects a curve in two or more distinct points is called a *secant*.

Let P be a fixed point of a plane curve, and P' a neighboring point. If P' be made to approach P along the curve, the secant PP' evidently approaches, in general, a definite limiting position, the line PT in the figure. The line thus approached is called the *tangent to the curve at* P, or is said to *touch the curve at* P. The point P is the *point of contact*.

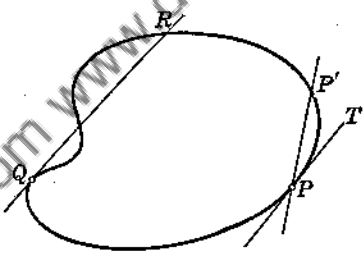

FIG. 62

The *slope of a curve* at any point is defined as the *slope of the tangent* at that point. Two curves intersecting in a point P are said *to be tangent at* P if they have the same slope — i.e. if they have a common tangent — at that point.

In elementary geometry a tangent to a circle is usually defined as a line which intersects the circle in one point. Such a definition would not hold for curves in general, as is shown by Fig. 61, where the line QR is both a secant and a tangent. (See also Ex. 19, p. 83, and Ex. 19, p. 94.) The definition given above holds in general.

EXERCISES

Remove the xy-term by rotation of axes, reduce the resulting equation to a standard form, and trace the curve on the new axes.

1. $xy = 2.$ **2.** $xy + 8 = 0.$

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4. $2x^2 + 2xy + 2y^2 = 3.$ *Ans.* $3x'^2 + y'^2 = 3.$

5. $x^2 + 6xy + y^2 + 8 = 0.$ *Ans.* $y'^2 - 2x'^2 = 4.$

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7. $7x^2 + 12xy - 2y^2 = 10.$ *Ans.* $2x'^2 - y'^2 = 2.$

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9. $19x^2 + 6xy + 11y^2 = 20.$ *Ans.* $2x'^2 + y'^2 = 2.$

10. $4xy - 3y^2 = 8.$ *Ans.* $x'^2 - 4y'^2 = 8.$

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12. $7x^2 - 12xy + 2y^2 = 44.$ *Ans.* $11y'^2 - 2x'^2 = 44.$

13. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0.$ *Ans.* $x'^2 - 2x' = 6y'.$

14. $11x^2 - 24xy + 4y^2 + 6x + 8y = 10.$ *Ans.* $4y'^2 - (x' - 1)^2 = 1.$

15. Represent $\tan(\theta + 45°)$ as a function of $\tan\theta$. (Expand by the addition formula; put $\tan\theta = x$, $\tan(\theta + 45°) = y$.)

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18. For the lever of Ex. 14, p. 42, if a weight of 6 lbs. is 4 ft. from the fulcrum, and if the lever weighs 12 oz. per ft., graph F as a function of l_2. What is the best length of lever to use? Does the graph have a meaning in the fourth quadrant?

19. Prove that if a line is parallel to the axis of a parabola, it intersects the curve in one and only one point.

20. Prove that the equation $x^{1/2} \pm y^{1/2} = \pm a^{1/2}$ represents a parabola, and trace the curve.

CHAPTER IX

TANGENTS AND NORMALS

59. Tangents to plane curves. A straight line that intersects a curve in two or more distinct points is called a *secant*.

Let P be a fixed point of a plane curve, and P' a neighboring point. If P' be made to approach P along the curve, the secant PP' evidently approaches, in general, a definite limiting position, the line PT in the figure. The line thus approached is called the *tangent to the curve at P*, or is said to *touch the curve at P*. The point P is the *point of contact*.

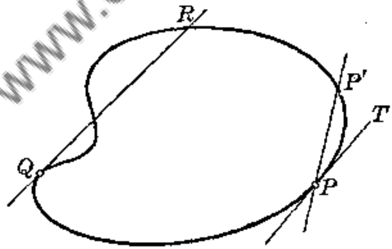

FIG. 62

The *slope of a curve* at any point is defined as the *slope of the tangent* at that point. Two curves intersecting in a point P are said to be *tangent at P* if they have the same slope — i.e. if they have a common tangent — at that point.

In elementary geometry a tangent to a circle is usually defined as a line which intersects the circle in one point. Such a definition would not hold for curves in general, as is shown by Fig. 61, where the line QR is both a secant and a tangent. (See also Ex. 19, p. 83, and Ex. 19, p. 94.) The definition given above holds in general.

A line perpendicular to the tangent at a point of a curve is called the *normal* to the curve at that point.

To determine the slope of a curve at a given point, and hence to find the equation of the tangent at that point, we have merely to *carry out analytically* the limit-process involved in the definition of tangent. In the next article the method will be applied to a problem, partly by way of illustration, and partly because the result — equation (5) of § 60 — is in itself a useful formula.

60. Tangent at a given point of the standard parabola. We proceed to find the equation of the tangent at any point* P: (x_1, y_1) on the parabola

(1) $$y^2 = 4ax.$$

Choose a point P' on the curve near the given point, and denote the distances PR, RP' by Δx, Δy respectively, so that the coördinates of P' are $(x_1 + \Delta x, y_1 + \Delta y)$. Since P' lies on the curve, its coördinates may be substituted for x and y in (1). This gives

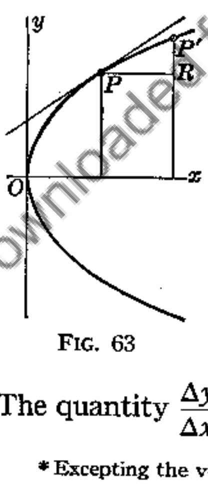

FIG. 63

(2) $$y_1^2 + 2y_1\Delta y + \overline{\Delta y^2} = 4ax_1 + 4a\Delta x.$$

Since (x_1, y_1) lies on the curve we have (§ 17)

(3) $$y_1^2 = 4ax_1,$$

whence (2) becomes

$$2y_1\Delta y + \overline{\Delta y^2} = 4a\Delta x,$$

or

(4) $$2y_1\frac{\Delta y}{\Delta x} + \frac{\Delta y}{\Delta x}\Delta y = 4a.$$

The quantity $\frac{\Delta y}{\Delta x}$ occurring here is the slope of the secant

* Excepting the vertex, where the tangent is evidently the line $x = 0$.

PP'. If now we let Δx approach 0, so that P' approaches P along the curve, Δy will also approach 0, but by § 59 *the slope m of the curve at P will be the limit of the ratio* $\frac{\Delta y}{\Delta x}$. Hence, when Δx approaches 0, (4) reduces to *

$$2y_1m = 4a, \qquad m = \frac{2a}{y_1}.$$

The desired equation is therefore (§ 23)

$$y - y_1 = \frac{2a}{y_1}(x - x_1),$$

or

$$y_1y - y_1^2 = 2ax - 2ax_1.$$

Simplifying by means of (3), we obtain the result:

The equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) on the curve is

$$(5) \qquad y_1y = 2ax + 2ax_1.$$

61. Summary. The general method employed in § 60 may be summarized as follows:

To find the slope m of a given curve at a point $P: (x_1\, y_1)$ on the curve, choose a neighboring point $P': (x_1 + \Delta x, y_1 + \Delta y)$ on the curve, substitute the coördinates of P' in the equation of the curve, and simplify. Divide through by Δx. Let Δx approach 0, $\frac{\Delta y}{\Delta x}$ at the same time approaching the value † m, and solve for m.

* In the term $\frac{\Delta y}{\Delta x}\Delta y$, the factor $\frac{\Delta y}{\Delta x}$ approaches m while the factor Δy approaches 0, so that the whole term approaches 0.

† Exceptionally, $\frac{\Delta y}{\Delta x}$ approaches no limit. In the case of the conics, this means that the tangent is the line through P parallel to Oy — i.e. the line $x = x_1$.

EXERCISES

Find the tangent to the curve at the given point.

1. $x^2 + y^2 = a^2$ at (x_1, y_1). *Ans.* $x_1x + y_1y = a^2$.
2. $x^2 - y^2 = a^2$ at (x_1, y_1). *Ans.* $x_1x - y_1y = a^2$.
3. $x^2 = 4ay$ at (x_1, y_1). *Ans.* $x_1x = 2ay + 2ay_1$.
4. $2xy = a^2$ at (x_1, y_1). *Ans.* $y_1x + x_1y = a^2$.
5. $x^2 - y^2 = 2ay$ at (x_1, y_1). *Ans.* $x_1x - y_1y = ay + ay_1$.
6. $x^2 + y^2 = 2ax$ at (x_1, y_1). *Ans.* $x_1x + y_1y = ax + ax_1$.
7. $x^2 - 2xy + y^2 + 2x + y - 6 = 0$ at $(2, 2)$. *Ans.* $2x + y = 6$.
8. $x^2 - xy + 3x - y = 1$ at $x = 2$. *Ans.* $4x - 3y + 1 = 0$.
9. $y = x^3 - 2x - 1$ at $x = 2$. *Ans.* $y = 10x - 17$.
10. $y = x^3 - 3x^2 + 2x - 5$ at $x = 1$. *Ans.* $x + y + 4 = 0$.

11. $x^2 + 2y^2 - 2x + 4y + 2 = 0$ at $(2, -1)$. *Ans.* $x = 2$.
12. By setting $m = 0$, find the highest and lowest points, and the center, of the ellipse $x^2 + 2xy + 2y^2 + 4x + 4y = 0$.
Ans. $(-4, 2)$, $(0, -2)$; $C : (-2, 0)$.
13. Solve Ex. 12 for the ellipse $x^2 - 3xy + 4y^2 - 3x + 7y - 2 = 0$.
Ans. $(3, 1)$, $(-\frac{15}{7}, -\frac{17}{7})$; $C : (\frac{3}{7}, -\frac{5}{7})$.
14. In Ex. 17, p. 94, after introducing the given data, clear the equation of fractions and find m. By setting $m = 0$, determine the most advantageous length for the lever. *Ans.* 5 ft. 9 in.
15. Apply the method suggested above (Ex. 14) in Ex. 18, p. 94. Why does the method fail in this case to give the most advantageous length?
16. A body moves in a vertical straight line, covering in time t a distance $x = 3t^2 - t^3$. What is the greatest height it will reach? *Ans.* 4 ft.

62. Two geometric properties of the parabola. Two important properties of the parabola are as follows.

THEOREM I: *The normal bisects the angle between the focal radius drawn to the point of contact and the line through that point parallel to the axis.*

THEOREM II: *The foot of the perpendicular from the focus upon any tangent lies on the tangent at the vertex.*

That is, in Fig. 64, the line PN bisects the angle FPQ, and the line FL intersects PT on the tangent at V.

The property embodied in Theorem I is the underlying principle of the "parabolic reflector," used in searchlights, spotlights, etc. Rays issuing from a point-source at F and striking the polished interior of a parabolic surface (§ 109) are reflected parallel to the axis of the surface.

The proof of the above theorems will be left to the student.

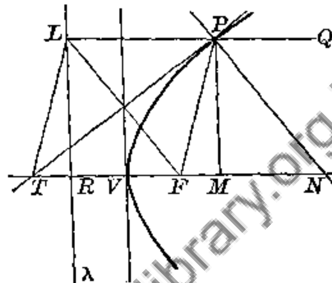

FIG. 64

EXERCISES

Using formula (5), § 60, in connection with Fig. 64, establish the following properties of the parabola.

1. The tangents at the ends of the latus rectum intersect on the directrix.

2. Any tangent intersects the directrix and the latus rectum (produced) in points equally distant from the focus.

3. The "subtangent" TM is bisected at the vertex — i.e. $TV = VM$.

4. The "subnormal" MN is constant and equal to RF, i.e. the distance from directrix to focus.

5. In Fig. 64, $FP = TF = FN$.

6. Prove Theorem I, § 62, geometrically.

7. Prove Theorem II, § 62, geometrically.

8. Prove Theorem I analytically, using § 10.

9. Prove Theorem I by finding the bisector of angle FPQ (§ 32).

10. Prove Theorem II analytically.

11. Prove that the tangent at any point (x_1, y_1) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.$$

12. Tangents are drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and to the circle $x^2 + y^2 = a^2$ at points having the same abscissa. Prove that these tangents cross Ox at the same point. (Ex. 11.)

13. Prove Ex. 1 for the ellipse. (Ex. 11.)

14. Prove that the tangents at the ends of the latera recta of an ellipse have slopes $\pm e$.

15. Prove that the perpendicular from a focus upon any tangent to an ellipse, and the line joining the center to the point of contact, intersect in a point on the directrix.

16. Prove that the product of the distances of the foci from any tangent to an ellipse is constant (equal to b^2).

Solve the following problems by ruler and compass.

17. Given a parabola and its vertex, construct the tangent and normal at any point. (Ex. 3.)

18. Given the axis, vertex, and one point of a parabola, construct the focus and directrix.

19. Given the axis and one point of a parabola, with the tangent at that point, construct the focus and directrix.

20. Given the axis, vertex, and one tangent to a parabola, construct the focus and directrix.

21. Solve Ex. 17 by a second method.

22. Solve Ex. 17 by a third method.

23. Solve Ex. 19 by a second method.

24. Solve Ex. 19 by a third method.

25. Given a parabola with its vertex, construct the tangent parallel to a given line.

26. Given an ellipse with its center, construct the tangent at any point. (Ex. 12.)

27. Prove that for all values of m (except $m = 0$) the line

$$y = mx + \frac{a}{m}$$

is tangent to the parabola $y^2 = 4ax$.

Solve the following by using the formula of **Ex. 27**.

28. Prove that only one tangent to a parabola can be drawn parallel to a given line, and none parallel to the axis.

29. Prove that the tangents drawn to a parabola from any point of the directrix are perpendicular.

30. State and prove the converse of the theorem of Ex. 29.

31. Solve Ex. 2 by a new method.

32. Prove Theorem II, § 62, by a new method.

33. Given a parabola with its axis, construct by ruler and compass the tangent parallel to a given line.

34. Prove that for all values of m the line

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

35. Using the formula of Ex. 34, prove the theorem of Ex. 2 for the ellipse.

36. Solve Ex. 16, using the formula of Ex. 34.

CHAPTER X

ALGEBRAIC CURVES

63. Algebraic curves. All plane curves other than the straight line (curve of first degree) and the conic sections (curves of second degree) are called *higher plane curves.* An *algebraic curve* is a curve whose Cartesian equation can be written as a polynomial in x and y, equated to 0. Algebraic curves of third degree are called *cubic curves,* or simply cubics; those of fourth degree, *quartics;* etc. In the present chapter we will consider some of the simpler types of higher algebraic curves, chiefly cubics and quartics. Factorable equations (§ 18) and equations having no locus will be excluded.

As we know, every equation of the second degree can be reduced to a "standard form," after which the curve is readily traced. On account of the great variety of possible forms, no similar process is feasible even for the cubics — much less so for curves of still higher degree. Instead, we shall try to discover, in each case, as many as possible of the algebraic properties and peculiarities of the equation, and then to translate these into geometric language. (A beginning in this direction was made in §§ 14–16.) While the line of attack depends more or less on the particular equation in hand, nevertheless the process can be systematized to some extent, as will now be shown.

64. Asymptotes. If the tangent to a curve *approaches a definite limiting position as its point of contact recedes indefinitely,* the line so approached is called an *asymptote.*

It can be shown that the asymptotes of a hyperbola as defined in § 50 also satisfy the above definition. While the hyperbola is the only conic having asymptotes, many higher plane curves possess one or more of these lines, and they play an important part in the geometry of those curves.

65. Behavior of y for large values of x; horizontal asymptotes. When x increases indefinitely in either direction, y of course may behave in a variety of ways. The two most frequently occurring situations are:

(*a*) y increases numerically without limit; or

(*b*) y approaches a definite limit a.

These results are easily interpreted. For example, in (*a*), if y becomes large and positive as x becomes large and positive, we know that the curve recedes indefinitely from both axes, in the first quadrant.

In (*b*), the curve approaches more and more closely the line $y = a$: this line is a *horizontal asymptote.**

66. Vertical asymptotes. It may happen that y increases indefinitely as x approaches some value b. This means that the curve approaches more and more closely the line $x = b$: this line is in general a *vertical asymptote.*

67. Restriction to definite regions. It is usually possible to determine certain definite portions of the plane within which the curve must lie. While no general directions can be given, in many cases we can *solve the equation for y* and note the *changes of sign* of the right member. The process will be explained presently by examples.

68. Summary. The preceding remarks may now be collected in the form of a definite sequence of steps:

* In rare instances a curve approaches a line more and more closely but the tangent approaches no limiting position, so that the line is not an asymptote. Such exceptions cannot occur among algebraic curves.

1. *Test the curve for symmetry.* (§ 16).

2. *Find the intersections with the axes.* (§ 14).

3. *Determine the behavior of y for large values of x. Find the horizontal asymptotes.* (§ 65.)

4. *Find the vertical asymptotes.* (§ 66.)

5. *Determine as narrowly as possible those regions of the plane in which the curve lies.* (§ 67.)

6. *Look for any further general information that may be obtainable; and if necessary, plot a few points.*

69. Polynomials. We take first the case in which y is a *polynomial* in x, also called a *rational integral function:*

$$y = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

where n is a positive integer and $a_0, a_1, \cdots, a_{n-1}, a_n$ are rational numbers. The cases $n = 1$, $n = 2$ have already been studied (§§ 28, 44); thus we may now assume $n \geqq 3$.

Before considering special examples, we shall apply our analysis to the polynomial in general, thus deducing certain results applicable to all curves of this class.

1. The curve cannot be symmetric about the x-axis.

2. The x-intercepts are the real roots of the polynomial.

3. As x becomes large in either direction, y becomes large (though not necessarily of the same sign as x).

4. This step may be omitted here, since no polynomial curve can have an asymptote, vertical or otherwise.

5. A polynomial can change sign only by passing through the value 0, hence the curve can cross the x-axis only by intersecting* it. The function actually will change sign at every x-intercept (root of the equation $y = 0$), except when the vanishing factor carries an *even exponent* (double root, quadruple root, etc.), in which case the curve will touch Ox and turn back. See Figs. 65, 66.

* A "discontinuous" curve may cross by jumping. See Fig. 66, p. 108.

6. General remark: For every value of x there is one and only one value of y. Hence the curve extends across the plane in an *unbroken arc from left to right*.

Example: Trace the curve

$$y = (x - 1)^2(x^2 - 4).$$

1. There is no symmetry.
2. Intercepts: $x = 0$, $y = -4$; $y = 0$, $x = 1, \pm 2$.
3. When x is large positive, y is large positive; x large negative, y large positive. The curve rises indefinitely in the first and second quadrants.
5. y changes sign as x passes through the values $-2, 2$. By step 3, at the extreme left $y > 0$; it therefore remains positive in the interval $x < -2$, becomes negative in the interval $-2 < x < 2$, positive for $x > 2$. The curve is restricted to the unshaded regions of the plane * in Fig. 65.
6. Plotting the additional point $(-1, -12)$, we draw the curve, with the x-scale four times the y-scale.

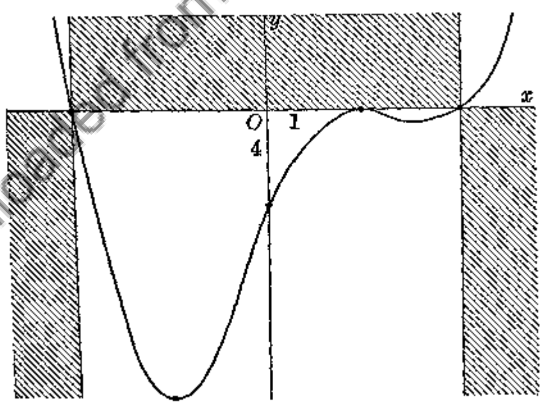

FIG. 65

* In the actual drawing of curves it is hardly desirable to shade the figure in this way. The device is adopted here to show visually the meaning of step 5.

EXERCISES

Trace the following curves, each on a suitable scale.

1. $y = x(x^2 - 1)$.
2. $y = x(4 - x^2)$.
3. $y = -x(x^2 - 6x + 5)$.
4. $y = x(x^2 + 5x + 6)$.
5. $y = (1 - x)^2(2 - x)$.
6. $y = x(x - 1)^2$.
7. $y = x^3 - 14x^2 + 59x - 70$.
8. $y = x^3 + 10x^2 + 29x + 20$.
9. $y = (x + 4)^3$.
10. $y = (1 - x)^3$.
11. $y = x^2(x^2 - 1)$.
12. $y = 1 - x^4$.
13. $y = 3x + x^2 - 3x^3 - x^4$.
14. $y = 4x^4 - 17x^2 + 4$.
15. $y = x(x - 2)^3$.
16. $y = 4x^3 - 3x^4$.
17. $y = x^4 - x^2 - 12$.
18. $y = x^4 - 3x^3 + 2x^2 - 6x$.
19. $y = (x - 1)^2(x^3 - 2x^2)$.
20. $y = x(x^2 - 1)(x - 2)^2$.

Draw the graphs of the following cubic functions.

21. The volume remaining when a slab of thickness 1 is cut from one face of a cube of edge l.

22. The volume remaining when slabs of thickness x, $2x$, $3x$ are cut from three mutually perpendicular faces of a cube of edge 1.

23. The volume of a box made by cutting squares of side x out of the corners of a piece of cardboard 6 in. square and turning up the sides.

24. Ex. 23 if the cardboard is 8×4 inches.

25. The volume of a right circular cylinder of radius x inscribed in a right circular cone of radius 4 and height 8.

26. The volume of a right circular cylinder of altitude $2y$ inscribed in a sphere of radius a.

27. The volume of a right circular cone of altitude y inscribed in a sphere of radius a.

28. Draw the curve from which $\cos 3\theta$ may be read off if $\cos\theta$ is given: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. (Put $x = \cos\theta$, $y = \cos 3\theta$.)

29. Draw the curve from which $\cos 4\theta$ may be read if $\cos\theta$ is given.

70. Rational fractions. Consider next the case in which y is equal to a *rational fraction:*

$$(1)\quad y = \frac{N}{D} = \frac{a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m}. \quad \begin{matrix}(n \geqq 0)\\(m \geqq 1)\end{matrix}$$

We assume that our fraction is in its lowest terms — i.e. that N and D contain no common factor.

We first analyze the general problem.

1. No symmetry with respect to Ox.

2. The x-intercepts are found by setting $N = 0$.

3. As x increases in either direction:

(*a*) If N is of higher degree than D, y becomes large, though not necessarily of the same sign as x. (See below.)

(*b*) If N is of lower degree than D, y approaches 0: the x-axis is an asymptote.

(*c*) If N and D are of the same degree, y approaches $\frac{a_0}{b_0}$: the line $y = \frac{a_0}{b_0}$ is an asymptote.

4. y becomes infinite as D approaches 0. Thus we find the real roots (if any) of the equation $D = 0$, say $r_1, r_2, \cdots$; the lines $x = r_1$, $x = r_2$, $\cdots$ are vertical asymptotes.

5. A fraction changes sign when either N or D does so. Thus we list the roots of $N = 0$, $D = 0$ (already found in steps 2 and 4), *casting out the roots of even order* (double, quadruple), and note for each of the others a change of sign of y and a passage of the curve across the x-axis: by intersection where $N = 0$, by jumping where $D = 0$.

6. There is one and only one value of y for every value of x, excepting those values for which $D = 0$.

Proof of 3(c): In this case, $m = n$, so that

$$y = \frac{a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n}{b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n}.$$

Divide numerator and denominator by x^n:

$$y = \frac{a_0 + \frac{a_1}{x} + \cdots + \frac{a_{n-1}}{x^{n-1}} + \frac{a_n}{x^n}}{b_0 + \frac{b_1}{x} + \cdots + \frac{b_{n-1}}{x^{n-1}} + \frac{b_n}{x^n}}.$$

Thus, when x increases indefinitely, y approaches $\frac{a_0}{b_0}$. Cases 3(*a*) and 3(*b*) may be handled similarly.

Example: Trace the cubic curve

$$y = \frac{x^2}{(x-1)(x-3)}. \tag{2}$$

1. There is no symmetry.
2. The only intersection with the axes is (0, 0).
3. By the above argument, as $x \to \pm\infty$, $y \to 1$: the line $y = 1$ is an asymptote. For large positive x, the denominator is less than the numerator and $y > 1$; for large negative x, $y < 1$. Thus the curve approaches the asymptote from above at the extreme right and from below at the extreme left. Finally, putting $y = 1$ in (2), we find $x = \frac{3}{4}$: the curve crosses the asymptote at $(\frac{3}{4}, 1)$. (It can be shown that *a curve of the nth degree may intersect an asymptote in not more than n − 2 points.*)
4. As $x \to 1$, and as $x \to 3$, y increases indefinitely: the lines $x = 1$, $x = 3$ are asymptotes.
5. The numerator vanishes at $x = 0$, but does not change sign because of the *even exponent*; the denominator, and hence the function, changes sign as x goes through 1, 3. This combined with step 3 limits the curve as shown.

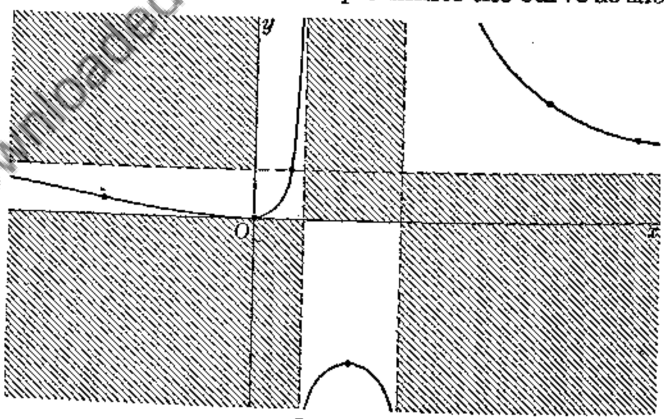

FIG. 66

EXERCISES

1. $y = \dfrac{2x}{1 + x^2}$.

2. $y = \dfrac{1}{1 + x^2}$.

3. $x^2y = 4y - 1$.

4. $x^2y = y - x$.

5. $x^3 - xy - 4x + y = 0$.

6. $x^3 - 4x^2 + xy + 2y = 0$.

7. $x^2y + x^2 - y - 4 = 0$.

8. $4x^2y - x^2 - 9y + 1 = 0$.

9. $2x^2y - x^2 + 3xy + 2x + y = 1$.

10. $x^2y - 3x^2 - y + 1 = 0$.

11. $y = \dfrac{1 - 2x + x^2}{x^2}$.

12. $y = \dfrac{1 + 2x - x^2 - 2x^3}{(x - 2)^2}$.

13. $y = \dfrac{4x^3 - 16x^2 - x + 4}{(x - 3)^2}$.

14. $y = \dfrac{3x^2 + 4x + 1}{x^2}$.

15. $y = \dfrac{(x - 1)(x - 2)^2}{x(x + 1)}$.

16. $y = \dfrac{x^3 - 8x^2 + 19x - 12}{4x^2 - 9}$.

17. $y = \dfrac{1 - 4x^2}{x(x^2 - 1)}$.

18. $y = \dfrac{x^3 + 1}{(x - 1)(x^2 - 4)}$.

19. $y = \dfrac{4x^3 - 13x^2 + 6x + 3}{x^2(x - 4)}$.

20. $y = \dfrac{(x^2 - 1)^2}{x^2(x + 4)}$.

21. $y = \dfrac{(x^2 - 2)^2}{x^2(x - 3)^2}$.

22. $y = \dfrac{2x^3 - 10}{x^3 - 3x^2 + 2x}$.

23. When does (1), § 70, represent a hyperbola?

24. A right circular cone is circumscribed about a sphere of radius a. Express the volume of the cone as a function of its altitude, and draw the graph. What kind of curve is this? *Ans.* $V = \frac{1}{3}\pi a^2 \cdot \dfrac{h^2}{h - 2a}$.

25. In Ex. 24, express the volume of the cone as a function of its radius, and draw the graph. *Ans.* $V = \frac{2}{3}\pi a \cdot \dfrac{r^4}{r^2 - a^2}$.

26. Draw the curve from which $\tan 2\theta$ may be read off when $\tan \theta$ is given. (Put $x = \tan \theta$, $y = \tan 2\theta$.)

27. Represent $\tan 3\theta$ as a function of $\tan \theta$: $\tan 3\theta = \dfrac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$. (Put $x = \tan \theta$, $y = \tan 3\theta$.)

28. Draw a curve from which $\sin 2\theta$ may be read off if $\tan \theta$ is given: $\sin 2\theta = \dfrac{2 \tan \theta}{1 + \tan^2 \theta}$. (Cf. Ex. 1.)

29. Draw a curve from which $\cos 2\theta$ may be read off if $\tan \theta$ is given.

30. Draw a curve from which $\sec 2\theta$ may be read off if $\cos \theta$ is given.

CHAPTER XI

POLAR COÖRDINATES

71. Distance and bearing. Instead of locating a point by its distances from two perpendicular lines (§ 2), we frequently, in ordinary usage, locate it by its distance and "bearing" from some fixed point: one town is 5 miles southeast of another; one boundary marker is 90 ft. N. 10° E. of another; etc. This alternative method, like the former one, has its counterpart in analytic geometry.

72. Polar coördinates. Let us choose a fixed line Ox in the coördinate plane, and a point O on this line. The position of any point P (Fig. 67) in the plane is determined if we know the length of the line OP together with the angle that this line makes with the fixed line Ox, both the distance and the angle being measured in a definite sense. The segment OP and the angle xOP are the *polar coördinates* of P; they are called the *radius vector* and the *polar angle* respectively, and are denoted by the letters r, θ. The fixed line Ox is the *initial line*, or *polar axis*, and the point O is the *pole*, or *origin*.

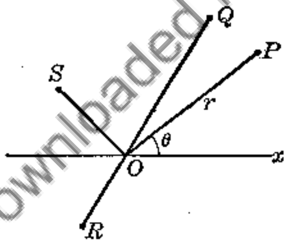

FIG. 67

The polar coördinates of a point are written in parentheses with the radius vector first, as P: (r, θ), or simply (r, θ). The polar angle is *positive* when measured *counterclockwise, negative clockwise;* the radius vector is *positive* if laid off *on the terminal side of* θ, *negative* if measured in the opposite direction, i.e. *on the terminal side produced through O.* Figure 67 shows the points $Q : (2, 60°)$, $R : (-1, 60°)$, $S : (-1, -\frac{1}{4}\pi)$.

To plot a point whose polar coördinates are given, it is best to begin by drawing the line on which the radius vector lies, i.e. the line making an angle θ with Ox, and then to lay off on that line, in the proper sense, the distance r.

73. The locus of an equation. If r and θ are connected by an equation of any form, we may assign values to θ and compute the corresponding value or values of r. The points thus determined all lie on a definite curve, the *locus* of the equation (cf. § 13). Curve plotting is done most conveniently on "polar coördinate paper," which is paper ruled in concentric circles and radial lines.

While to every pair of polar coördinates corresponds a single definite point, the converse is not true: the same point may be represented by various pairs of coördinates. Thus, in Fig. 67, the coördinates (2, 60°), (− 2, 240°), (− 2, − 120°), (1, − 300°) all represent the point Q.

To each equation corresponds a single definite curve, but the fact that a given point may be represented by different pairs of coördinates makes it possible that a curve may be represented in the polar system by more than one equation. Thus the equations $r = 2$, $r = -2$ represent the same curve, a circle of radius 2 with center at the origin.

74. One-valued functions. We take up the problem of tracing polar curves, considering first the case in which there is *one value of r for each value of θ.*

Examples: (*a*) Trace the curve $r = 2a \cos \theta$.

θ	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$
r	$2a$	$\sqrt{3}a$	$\sqrt{2}a$	a	0

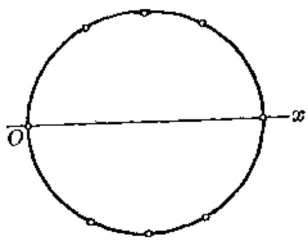

FIG. 68

Plotting these points, we obtain the upper half of the curve (Fig. 68.) Since for values of θ in the *second* quadrant $\cos\theta$, and hence r, is negative, the curve falls in the *fourth* quadrant; the values of $\cos\theta$ are numerically the same as in the first quadrant, but in reverse order, so that the lower half of the curve is symmetric to the upper half. In the third quadrant $\cos\theta$ takes the same values as in the first, and in the same order, but negative; the upper half is repeated. Similarly, for θ in the fourth quadrant, the lower half is repeated. Since a complete period of the cosine function has now been covered, further values of θ will merely repeat the same curve.

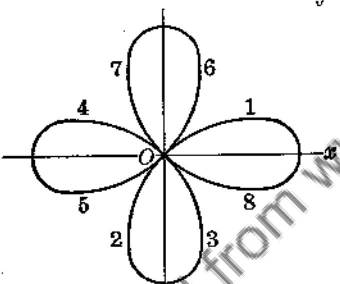

Fig. 69

(*b*) Trace the *four-leaved rose* $r = a\cos 2\theta$.

θ	$0°$	$15°$	$22\frac{1}{2}°$	$30°$	$45°$
2θ	$0°$	$30°$	$45°$	$60°$	$90°$
r	a	$\frac{1}{2}\sqrt{3}a$	$\frac{1}{2}\sqrt{2}a$	$\frac{1}{2}a$	0

Plotting the points found above,* we obtain the half-loop numbered 1. As θ ranges from 45° to 90°, 2θ ranges from 90° to 180°: thus the values of $\cos 2\theta$ are numerically the same as those found above, but in reverse order and negative. Since r is negative, the corresponding portion of the curve lies not in the first, but in the third quadrant: the half-loop 2. Reflecting in Oy, we obtain the arcs 3, 4; reflecting in Ox, the balance of the curve. Since this covers a complete period of the function $\cos 2\theta$, we have the entire curve.

* It is of course the values of r and θ, not r and 2θ, that are plotted. Thus the second point is $(\frac{1}{2}\sqrt{3}a, 15°)$, etc.

EXERCISES

Trace the following curves on polar coördinate paper.

1. $r = a \cos^2 \theta$.
2. $r = a \sin^2 \theta$.
3. $r = a(1 + \sin^2 \theta)$.
4. $r = a(2 - \cos^2 \theta)$.
5. $r = a(\cos \theta - \sin \theta)$.
6. $r = a \sin 2\theta$.
7. $r(2 - \sin \theta) = a$ (*ellipse*).
8. $r(2 + \cos \theta) = a$.
9. $r = a(3 + \cos \theta)$ (*limaçon*).
10. $r = a(2 - \sin \theta)$.
11. $r = a(1 - \sin \theta)$ (*cardioid*).
12. $r = a(1 + \cos \theta)$.
13. $r = a(2 \cos \theta - 1)$ (*limaçon*).
14. $r = a(1 + 2 \sin \theta)$.
15. $r(1 + \sin \theta) = 1$.
16. $r(1 - \cos \theta) = 1$.
17. $r = 2a \cos \theta \cot \theta$.
18. $r = 2a \sin \theta \tan \theta$.
19. $r = a \sin 3\theta$.
20. $r = a \cos 3\theta$.
21. $r = a \sin^2 2\theta$.
22. $r = a \cos^2 2\theta$.
23. $r = a \tan \theta$.
24. $r = a \sec \theta$.
25. $r = a(1 - \sin 2\theta)$.
26. $r = a(1 - 2 \cos 2\theta)$.

75. Two-valued functions. Consider the case in which r^2 is expressed as a function of θ, so that there are *two values of r for each value of θ.*

Example: Trace the *lemniscate* $r^2 = a^2 \cos 2\theta$.

θ	0	15°	$22°\frac{1}{2}$	30°	45°
2θ	0	30°	45°	60°	90°
r^2	a^2	$0.87a^2$	$0.71a^2$	$0.5a^2$	0
r	$\pm a$	$\pm 0.93a$	$\pm 0.84a$	$\pm 0.71a$	0

Plotting these points, we get the arcs 1, 2. As θ ranges from 45° to 135°, 2θ ranges from 90° to 270°, $\cos 2\theta$ is *negative* and r *imaginary*. For $135° < \theta < 180°$, we have $270° < 2\theta < 360°$, and r assumes the same sequence of values as above (in reverse order) — the arcs 3, 4. For $180° < \theta < 360°$ we merely repeat the same curve.

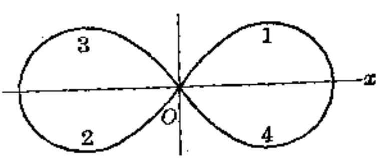

FIG. 70

EXERCISES

Trace the following curves.

1. $r^2 = a^2 \sin\theta$.
2. $r^2 = a^2 \cos\theta$.
3. $r^2 = a^2 \sin 2\theta$.
4. $r^2 = 1 + \sin^2\theta$.
5. $r^2(1 + \cos^2\theta) = a^2$.
6. $r^2(1 + 3\sin^2\theta) = a^2$.
7. $r^2 \sin 2\theta = a^2$.
8. $r^2 \cos 2\theta = a^2$.
9. $r^2 = 3 - 4\cos^2\theta$.
10. $r^2 = 1 - 4\sin^2\theta$.
11. $r^2 = a^2(1 + \cos\theta)^3$.
12. $r^2 = a^2(1 - 2\sin\theta)^3$.
13. $r^2 = a^2(2 - \cos\theta)$.
14. $r^2 = a^2(\sin\theta + 4)$.
15. $r^2 = a^2(1 + \sin\theta)$.
16. $r^2 = a^2(2\cos\theta - 1)$.
17. $r^2 = a^2 \sin\theta(1 + \sin\theta)$.
18. $r^2 = a^2 \cos\theta(1 - \cos\theta)$.
19. $r^2 = a^2(\sin\theta + \cos\theta)$.
20. $r^2 = a^2 \sin\theta(1 - 2\sin\theta)$.
21. $r^2 = a^2 \cos 3\theta$.
22. $r^2 = a^2 \cos\theta \cos 2\theta$.

76. Transformation from one system to the other. The usefulness of polar coördinates lies chiefly in the fact that many important curves are more easily traced, and their properties more easily developed, by using the polar equation of the curve. Not infrequently both forms are useful for the same curve, some properties appearing more readily from the Cartesian, others from the polar form. This suggests the desirability of formulas enabling us to pass from either system to the other.

Let the point P have the Cartesian coördinates x, y and the polar coördinates r, θ. Then it is obvious that

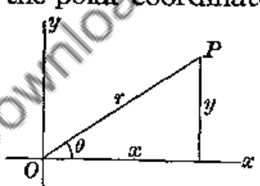

FIG. 71

(1) $$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases}$$

and

(2) $$\begin{cases} r^2 = x^2 + y^2, \\ \tan\theta = \dfrac{y}{x}. \end{cases}$$

Examples: (*a*) Obtain the equation of the circle

$$x^2 + y^2 = 2ax$$

in polar coördinates.

Since $x^2 + y^2 = r^2$, we have at once

$$r^2 = 2ar \cos \theta, \text{ or } r = 2a \cos \theta.$$

(*b*) Change the equation $r^2 = a^2 \sin \theta$ to Cartesian coördinates.

While not necessary, it is convenient first to multiply both members by r, to introduce the combination $r \sin \theta$, and then avoid radicals by squaring:

$$r^3 = a^2 r \sin \theta, \qquad r^6 = a^4 (r \sin \theta)^2.$$

This gives at once

$$(x^2 + y^2)^3 = a^4 y^2.$$

EXERCISES

Assume that the axes are placed as in Fig. 71.

1. Find the Cartesian coördinates of the points (*a*) $(1, 30°)$; (*b*) $(-1, \frac{1}{2}\pi)$; (*c*) $(-2, 0)$; (*d*) $(3, -60°)$; (*e*) $(-2, \frac{5}{8}\pi)$; (*f*) $(3, 225°)$.

2. Find the polar coördinates of the points (*a*) $(2, 2)$; (*b*) $(\sqrt{3}, -1)$; (*c*) $(0, 0)$; (*d*) $(-3, -4)$; (*e*) $(-5, -5)$; (*f*) $(0, -3)$; (*g*) $(-1, 0)$.

Find the equations of the following curves in polar coördinates.

3. $y = x$.

4. $x^2 + y^2 = a^2$.

5. $y = 2x + 3$.

6. $x + y = 1$.

7. $y = a$.

8. $x = a$.

9. $x^2 = 4ay$.

10. $y^2 = 4a(x - a)$.

11. $(x^2 + y^2)^2 = ay^3$.

12. $(x^2 + y^2)^2 = 8x$.

13. $y = x^3$.

14. $y^2 = x^3$.

15. $x \cos \beta + y \sin \beta = p$. (§ 31.) *Ans.* $r \cos (\theta - \beta) = p$.

16. $x^4 - y^4 = 2a^2 xy$. *Ans.* $r^2 = a^2 \tan 2\theta$.

Find the equations of the following curves in Cartesian coördinates.

17. $r = a$.

18. $\theta = 45°$.

19. $\theta = 0$.

20. $\tan \theta = 2$.

21. $\sin \theta = \frac{3}{5}$.

22. $\cos \theta + \frac{4}{5} = 0$.

23. $r = 2 \csc \theta$.

24. $r + 3 \cos \theta = 0$.

25. $r = \sec \theta \tan \theta$.

26. $r^2 \cos 2\theta = a^2$.

27. $r \cos (\theta - \frac{1}{4}\pi) = \sqrt{2}$. *Ans.* $x + y = 2$.

28. $r = a(1 - \cos \theta)$. *Ans.* $(x^2 + y^2 + ax)^2 = a^2(x^2 + y^2)$.

77. Locus problems in polar coördinates. To find the equation of a locus, using polar coördinates, the procedure is exactly the same as in §§ 20, 21. That is, we *assume a point P: (r, θ) in a general position on the curve*, and from the statement of the problem or by means of some characteristic property of the figure try to obtain an equation involving r, θ, and the constants of the problem: this equation must be the equation of the given locus.

Polar coördinates are strongly indicated in that numerous class of problems where the distance of the moving point from a fixed point varies according to some simple law. The fixed point should usually be taken as pole, and a line of symmetry (if such exists) as polar axis.

Example: In Fig. 72, a random line is drawn through O intersecting the circle at P; P is projected to M; a length $OQ = OM$ is laid off on OP. Find the locus of Q.

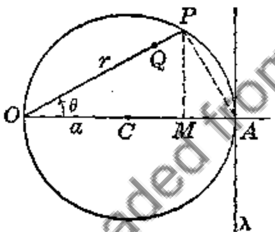

FIG. 72

By trigonometry,

$$OP = OA\cos\theta = 2a\cos\theta,$$

$$OM = OP\cos\theta = 2a\cos^2\theta;$$

since

$$OM = OQ = r,$$

$$r = 2a\cos^2\theta. \qquad \text{(Ex. 1, p. 113.)}$$

EXERCISES

Find the polar equations of the following loci. Draw the figures.

1. A circle of radius 5 with center at O.
2. A line through O making an angle of 45° with Ox.
3. A circle of radius a with center at $(a, 0)$. (Fig. 68, p. 112.)
4. A circle of radius a with center at $(a, \frac{1}{2}\pi)$.
5. A line through $(a, 0)$ perpendicular to Ox.
6. A line through $(a, \frac{1}{2}\pi)$ parallel to Ox.

7. The path of a point which is equidistant from a given point and a given line. Take the given point as pole and the given line perpendicular to Ox at $(a, 0)$. *Ans.* $r(1 + \cos\theta) = a$.

8. Ex. 7, if the moving point is half as far from the given point as from the given line. *Ans.* $r(2 + \cos\theta) = a$.

9. Ex. 7, if the moving point is twice as far from the given point as from the given line. *Ans.* $r(1 + 2\cos\theta) = a$.

10. Ex. 7, taking the given line parallel to Ox through $(a, \frac{1}{2}\pi)$. *Ans.* $r(1 + \sin\theta) = a$.

11. In Fig. 72, a distance $OR = MA$ is laid off on OP. Find the locus of R. (Ex. 2, p. 113.)

12. In Fig. 72, a distance $OS = OM - MA$ is laid off on OP. Show that the locus of S is a "four-leaved rose." (Fig. 69, p. 112.)

13. In Fig. 72, the line OP is produced to a point R such that $PR = 2a$. Show that the locus of R is a cardioid. (Ex. 12, p. 113.)

14. In Fig. 72, a point R is marked on OP such that $OR = OP - OM$. Find the locus of R. (Ex. 27, p. 113.)

15. In Fig. 72, a point R is marked on OP such that $OR = MP$. Show that the locus of R is a four-leaved rose. (Ex. 6, p. 113.)

16. In Fig. 72, let OP produced intersect λ at a point T: find the locus of the midpoint of PT, and trace the curve. *Ans.* $r = a(\sec\theta + \cos\theta)$.

17. The center of a circle moves along Ox; tangents to the circle are drawn through O. Find the locus of the points of tangency.

18. A tangent drawn to a circle with center at O intersects the axes at A, B. Find the locus of the midpoint of AB. *Ans.* $r\sin 2\theta = a$.

19. Find the equation of a circle of radius a with center at (r_1, θ_1). *Ans.* $r^2 - 2r_1r\cos(\theta - \theta_1) = a^2 - r_1^2$.

20. A point moves so that the product of its distances from two fixed points is one-fourth the square of the distance between the points. Using Cartesian coördinates, and taking the points $(\pm b, 0)$, find the equation of the locus. Show by transformation that the locus is a lemniscate (p. 113).

CHAPTER XII

PARAMETRIC EQUATIONS

78. Parametric equations. For many loci, it is difficult to obtain either the Cartesian or the polar equation directly; instead, the definition leads naturally to *two* equations giving x and y respectively in terms of some third variable. This auxiliary variable is called a *parameter*, and the two equations are *parametric equations* of the curve. *To obtain the ordinary Cartesian equation* we have merely to *eliminate the parameter between the two equations.*

The Cartesian equation of a curve is unique: for a given curve in a given position on the axes there can be only one equation.* As regards parametric representation, the case is quite otherwise: two different problems may lead, each in a perfectly natural way, to two different pairs of parametric equations for the same curve. For instance, if a point moves in a plane in a certain way, the coördinates of the point at any time t may be given by the equations

(1) $$x = t, \qquad y = 1 - t.$$

Under another law of motion the equations may be

(2) $$x = \cos^2 t, \qquad y = \sin^2 t.$$

Adding either equations (1) or equations (2), we get

(3) $$x + y = 1,$$

which shows that in each case the point moves in the straight line (3).

Sometimes the parametric equations are merely inci-

* While it is possible to find fault with this remark, the objections are purely artificial: for all practical purposes, the statement is strictly true. See the footnote, p. 36.

dental, being discarded entirely as soon as the Cartesian equation is obtained. In other cases the Cartesian equation is so complicated that the problem is best studied by means of the parametric equations. Again, it may happen that both forms are useful. Thus, in the first motion-problem above, it appears from (3) that the point moves in a straight line. Since, by (1), both x and y are *linear functions* of t, it follows (§ 29) that the point moves with constant speed — a fact which is not shown at all by (3).

79. Representation of a portion of a curve. In the elementary applications the parameter is of course restricted to real values, and frequently, by the very nature of the problem, is still further restricted, for example to positive values. (Compare the last paragraph of § 28.) Because of these restrictions, the parametric equations in many cases represent only a portion of the curve whose Cartesian equation is found by elimination.* Thus, if t may assume any real value, equations (1) of § 78 represent the entire line; equations (2), only the segment in the first quadrant, since x and y must be positive. In studying the motion of a point, we would as a rule consider the motion as starting at time $t = 0$, and would be interested only in subsequent — i.e. positive — values of t; if so, equations (1) represent only the part of the line to the right of the y-axis, since x must be positive.

80. Point-plotting from parametric equations. To plot, by points, a curve represented by parametric equations, we merely assign suitable values to the parameter and compute the corresponding values of x and y.

* Such a phenomenon is not surprising. If we prove that every (x, y)-pair satisfying two parametric equations also satisfies a certain Cartesian equation, it does not follow, conversely, that every pair satisfying the Cartesian equation must satisfy the parametric equations.

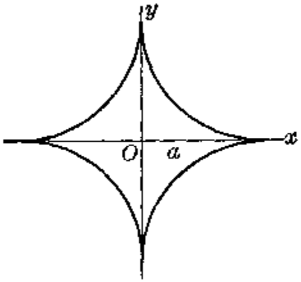

FIG. 73

Example: Plot the curve

(1) $x = a\cos^3\theta, \qquad y = a\sin^3\theta.$

θ	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$
x	a	$\frac{3}{8}\sqrt{3}a$	$\frac{1}{4}\sqrt{2}a$	$\frac{1}{8}a$	0
y	0	$\frac{1}{8}a$	$\frac{1}{4}\sqrt{2}a$	$\frac{3}{8}\sqrt{3}a$	a

Plotting these points, we get the portion of Fig. 73 lying in the first quadrant. Upon taking the cube root of the square of each member, the given equations become

$$x^{2/3} = a^{2/3}\cos^2\theta, \qquad y^{2/3} = a^{2/3}\sin^2\theta;$$

adding, we get the Cartesian equation

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

From this it appears (Theorem I, § 16) that the curve is symmetric to both axes, whence the balance of the curve may be obtained by reflection.

The student is warned not to confuse parametric representation with polar coördinates. The parameter θ here occurring is quite different from the polar angle θ. To make sure of this, divide the second of equations (1) by the first, member by member:

$$\tan^3\theta = \frac{y}{x}, \qquad \tan\theta = \frac{y^{1/3}}{x^{1/3}}.$$

For the polar θ, we have (§ 76)

$$\tan\theta = \frac{y}{x}.$$

EXERCISES

Find the Cartesian equation of the given curve, and draw the curve from the Cartesian equation. In case the Cartesian and parametric representations are not fully equivalent, determine what part of the curve is represented by the given equations.

1. $x = 2 - 3t,\ y = 2 - 6t.$
2. $x = t^2 - 1,\ y = 2t^2 - 4.$
3. $x = 2\cos^2\theta,\ y = 2\cos 2\theta.$
4. $x = \log_{10} 10n,\ y = \log_{10} n^2.$
5. $x = 10^{-t^2},\ y = 2 \cdot 10^{-t^2} - 2.$
6. $x = \sec^2\phi,\ y = 2\tan^2\phi.$
7. $x = a\sin 2\phi,\ y = a\cos 2\phi.$
8. $x = 4\sin\phi,\ y = 3\cos\phi.$
9. $x = \sin\theta,\ y = \cos 2\theta.$
10. $x = 2\cos 2\theta,\ y = \cos\theta.$
11. $x = t^2,\ y = 1 - t.$
12. $x = t + 2,\ y = t^2 - 1.$
13. $x = \frac{1}{2}\sin 2\alpha,\ y = \sin^2\alpha.$
14. $x = \cos^2\alpha,\ y = \sin 2\alpha.$
15. $x = a\sec\phi,\ y = b\tan\phi.$
16. $x = a\cot\alpha,\ y = a\csc\alpha.$
17. $x = \sin\phi + \cos\phi,\ y = \sin\phi - \cos\phi.$ *Ans.* $x^2 + y^2 = 2.$

Plot the curve (or part-curve) by points, using the parametric equations (§ 80); obtain the Cartesian equation.

18. $x = 1 + t^2,\ y = 4t - t^3.$ *Ans.* $y^2 = (x - 1)(x - 5)^2.$
19. $x = 1 - t^2,\ y = t + t^3.$ *Ans.* $y^2 = (1 - x)(2 - x)^2.$
20. $x = \sin\phi + \cos\phi,\ y = \sin\phi.$ *Ans.* $x^2 - 2xy + 2y^2 = 1.$
21. $x = \sin\theta,\ y = \cos 3\theta.$ *Ans.* $y^2 = (1 - x^2)(1 - 4x^2)^2.$
22. $x = \dfrac{1}{1 + t},\ y = \dfrac{2}{1 + t^2}.$ *Ans.* $y = \dfrac{2x^2}{2x^2 - 2x + 1}.$
23. $x = \dfrac{1}{1 + t},\ y = \dfrac{1}{1 - t^2}.$ *Ans.* $x^2 - 2xy + y = 0.$
24. $x = a\cos^4\theta,\ y = a\sin^4\theta.$ (Cf. Ex. 20, p. 94.)

25. Find the parametric equations of the straight line in terms of the parameter $k = \dfrac{P_1P}{P_1P_2}$, where $P : (x, y)$ is any point on the line, and $P_1 : (x_1, y_1)$, $P_2 : (x_2, y_2)$ are two given points on the line. (See § 6.)
Ans. $x = x_1 + k(x_2 - x_1),\ y = y_1 + k(y_2 - y_1).$

26. A circle is drawn on the major axis of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ as a diameter (Fig. 74); the ordinate MP of any point P is produced to Q on the circle; the line OQ is drawn, making with Ox an angle ϕ (called the *eccentric angle corresponding to the point* P). Find the parametric equations of the ellipse in terms of the eccentric angle.
Ans. $x = a\cos\phi,\ y = b\sin\phi.$

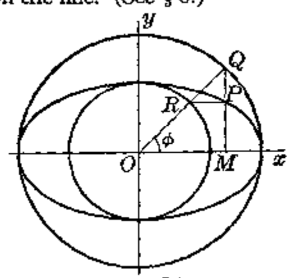

FIG. 74

81. Motion in a plane curve. As an important application of parametric representation we cite the following problem of mechanics.

When a point moves in a plane curve under the action of a given force* or system of forces, an especially convenient way of studying the motion is to express the Cartesian coördinates of the point as functions of the time t. The equations giving x and y in terms of t are *parametric equations of the path;* upon elimination of t the Cartesian equation is obtained. (Compare the remarks of §§ 78–79.)

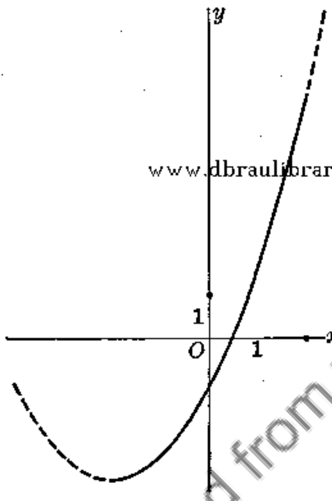

FIG. 75

Example: Determine the path of a point moving according to the laws†

$$x = \cos t,$$
$$y = 4\cos t + \cos 2t.$$

Since $\cos 2t = 2\cos^2 t - 1$, we have

$$y = 4\cos t + 2\cos^2 t - 1$$
$$= 4x + 2x^2 - 1.$$

This equation in standard form (§ 43) is

$$(x + 1)^2 = \tfrac{1}{2}(y + 3),$$

representing a parabola with vertex at $(-1, -3)$, opening upward. When $t = 0$, $x = 1$, $y = 5$. As $\cos t$ ranges from 1 to -1, the point moves (along the parab-

* The "point" is supposed to be endowed with mass — a "material particle." Further, the argument applies to a body of any size or shape, provided that, for present purposes, the motion of the entire body is completely characterized by the motion of one of its points. This would be the case, for instance, in computing the range of a projectile, or determining the orbit of a planet.

† When trigonometric functions of the time t occur, as they do in a great many important problems, the beginner may be somewhat puzzled by the interpretation of "time" as an *angle.* The notation means that the value of t (in seconds) is substituted in the formula and the result interpreted as if the angle were measured in *radians.* Thus in the present instance, when $t = 1$ (sec.), $x = \cos 1$ (rad.) $= 0.54$; when $x = 0$, $\cos t = 0$, $t = \frac{1}{2}\pi = 1.57$ sec.

ola) to $(-1, -3)$; it then returns to the starting point, and subsequently repeats the same cycle indefinitely.

EXERCISES

A point moves in a plane curve according to the given laws. Find the Cartesian equation of its path, and trace the motion from time $t = 0$.

1. $x = 1 + t^2,\ y = -2t^2$.
2. $x = \cos^2 t,\ y = -\sin^2 t$.
3. $x = t,\ y = 3t^2 + 6t$.
4. $x = t^2,\ y = t - t^2$.
5. $x = a \cos 3t,\ y = a \sin 3t$.
6. $x = a \cos 2t,\ y = b \sin 2t$.
7. $x = \sin t,\ y = \cos 2t$.
8. $x = 1 - \cos t,\ y = \cos 2t$.
9. $x = \sin t,\ y = \sin 2t$.
10. $x = \sin t + \cos t,\ y = \sin t$.
11. $x = \sin t,\ y = \sin 3t$.
12. $x = \cos t,\ y = \cos 3t$.
13. $x = \sin t + \cos t,\ y = \cos 2t$. *Ans.* $y^2 = x^2(2 - x^2)$.
14. $x = \cos 2t,\ y = \cos 3t$. *Ans.* $2y^2 = (1 + x)(1 - 2x)^2$.
15. $x = \sin t + \cos t,\ y = \frac{1}{2} \sin 4t$. *Ans.* $y^2 = x^2(2 - x^2)(x^2 - 1)^2$.

SOLID ANALYTIC GEOMETRY

CHAPTER XIII

COÖRDINATES IN SPACE

82. Rectangular Cartesian coördinates. To fix the position of a point in three-dimensional space, it is obvious that three magnitudes must be given. Thus to locate a point in the interior of a room, we may give its height above the floor and its distances from two adjacent walls.

Usually the simplest and most convenient method of determining the position of a point in space is *by means of its distances from three mutually perpendicular planes,* as in the example just cited. These distances are the *rectangular Cartesian coördinates* of the point; the three planes are the *coördinate planes,* their three lines of intersection are the *coördinate axes,* and their point of intersection is the *origin.* The coördinates are denoted by the letters x, y, z, and are written $P : (x, y, z)$, or merely (x, y, z). The three axes are called the x-axis, the y-axis, and the z-axis; the three planes are the xy-plane (containing the x- and y-axes), the yz-plane, and the zx-plane.

Evidently a definite positive sense must be chosen for each coördinate; that is, the coördinates are *directed segments,* as in plane geometry.

Space is divided by the coördinate planes into eight compartments, or *octants.* The region in which all three coördinates are positive is called the *first octant;* there will be no occasion to refer to the others by number.

83. Figures. In the system of drawing adopted in this book, *parallel lines are represented by parallel lines:* i.e. if two lines in space are parallel, they are shown in the figure by lines that are actually parallel, instead of by lines that run to a "vanishing point."

Two of the axes are represented by perpendicular lines, while the third, which of course is supposed to be perpendicular to the other two, is shown by a line drawn in any convenient direction. The axes are usually placed as in Fig. 76 (with a different arrangement permissible, of course, when more convenient), the positive half of each axis being the part drawn in full. Figures in the *yz*-plane, or in a plane parallel to that plane, are drawn in their true form and proportions, but all others are distorted, due to foreshortening in the direction of *x*.

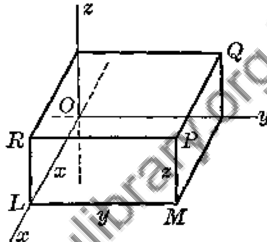

FIG. 76

In Fig. 76, the coördinates of P are $OL = QP = x$, $LM = RP = y$, $MP = z$.

84. Distance between two points. Given any two points $P_1 : (x_1, y_1, z_1)$, $P_2 : (x_2, y_2, z_2)$, we note that, in Fig. 77,

$$LM = x_2 - x_1,$$
$$NQ = y_2 - y_1,$$
$$RP_2 = z_2 - z_1.$$

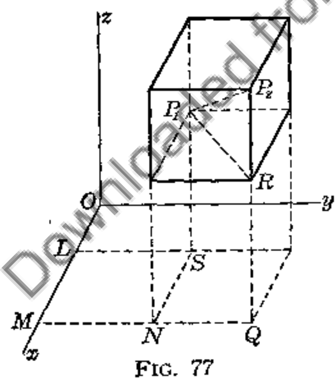

FIG. 77

Since

$$\overline{P_1P_2} = \sqrt{\overline{P_1R}^2 + \overline{RP_2}^2} = \sqrt{\overline{SN}^2 + \overline{NQ}^2 + \overline{RP_2}^2},$$

the length of the segment P_1P_2 is

$$(1) \qquad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

EXERCISES

1. Plot the points (2, 4, 1), (3, − 1, 2), (− 2, 2, − 3), (0, 1, − 1).

2. Plot the points (3, 2, 2), (− 2, 4, 1), (3, − 1, − 3), (2, − 3, 0).

3. Through the point (2, 3, 4) draw lines intersecting each of the coördinate axes at right angles.

4. In each coördinate plane, draw a line through O making an angle of 45° with each of the axes, and a perpendicular to this line through an arbitrary point in that plane.

5. Draw a rectangular parallelepiped, or *box*, with its edges parallel to the axes and having the points (3, 2, 1), (4, 4, 2) as ends of a diagonal.

6. In each coördinate plane, draw a circle of radius a with center at O.

7. In the zx-plane, plot the parabola $z^2 = 4ax$.

8. In the xy-plane, plot the parabola $x^2 = 4ay$.

9. What is the distance of the point (x, y, z) from Ox? From Oy? From Oz? From O?

10. Where is a point situated if

(*a*) $x = 0$?	(*b*) $z = 0$?	(*c*) $x = y = 0$?
(*d*) $y = z = 0$?	(*e*) $x = 2$?	(*f*) $x = 2, y = 1$?
(*g*) $x = z$?	(*h*) $y = z, x = 0$?	(*i*) $x = y = z$?

Find the distances between the following pairs of points.

11. (3, − 1, 2), (5, − 2, − 3). **12.** (1, 7, − 4), (− 3, − 2, − 2).

13. $(\frac{3}{2}, 0, -\frac{2}{3}), (\frac{1}{2}, 2, -\frac{1}{3})$. **14.** $(1, -2, -\frac{3}{2}), (-\frac{1}{2}, 0, \frac{5}{2})$.

15. (*a*) Prove that the lines joining (6, − 9, 10), (1, 1, − 5), (6, 11, 0) form a right triangle; (*b*) find its area. *Ans.* (*b*) $25\sqrt{21}$.

16. (*a*) Prove that the lines joining (3, 5, 1), (2, 3, − 2), (6, 1, − 2) form a right triangle; (*b*) find its area. *Ans.* (*b*) $\sqrt{70}$.

17. (*a*) Prove that the lines joining (6, 2, 4), (2, 0, − 2), (4, − 2, 10) form an isosceles triangle; (*b*) find its area. *Ans.* (*b*) $6\sqrt{19}$.

18. (*a*) Prove that the lines joining (6, − 2, 3), (1, 3, 2), (0, 2, − 5) form an isosceles triangle; (*b*) find its area. *Ans.* (*b*) $\sqrt{638}$.

19. Prove that the points (4, 1, 2), (2, 1, 1), (3, 3, − 1), (5, 3, 0) are the vertices of a rectangle.

20. A point is at the distance $\sqrt{33}$ from Ox, $2\sqrt{7}$ from Oy, $\sqrt{11}$ from Oz. Find its coördinates. *Ans.* $(\pm\sqrt{3}, \pm 2\sqrt{2}, \pm 5)$.

21. A moving point is always equidistant from the points (2, 3, 1), (1, 2, 0). Find the equation of its locus. *Ans.* $2x + 2y + 2z = 9$.

22. A point moves always at the distance 3 from (1, − 2, 2). Find the equation of its locus. *Ans.* $x^2 + y^2 + z^2 - 2x + 4y - 4z = 0$.

85. Direction angles; direction cosines. Given any directed line λ passing through the origin, the angles α, β, γ formed by this line with the positive x-, y-, and z-axes are called the *direction angles* of the line, and the cosines of these angles are the *direction cosines* of the line. Direction cosines are denoted by l, m, n: that is,

$$l = \cos\alpha, \qquad m = \cos\beta, \qquad n = \cos\gamma.$$

More generally, if the given line does not pass through the origin, its direction angles and direction cosines are defined as *equal to those of the parallel line through the origin.*

Given the three direction cosines l, m, n of any directed line λ', let $P:(x, y, z)$ be any point on the parallel λ through the origin, and denote the distance OP by ρ. Evidently

$$x = OP\cos\alpha = l\rho;$$

similarly,

$$y = m\rho, \qquad z = n\rho.$$

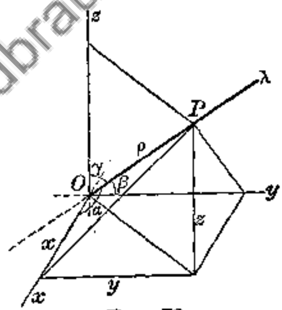

FIG. 78

Thus the coördinates of P can be found if l, m, n are known, so that the line λ is determined. It follows that *the direction of any line is determined if its direction cosines are given.*

If the positive sense on the line be reversed, its direction angles are replaced by their supplements, and the signs of the direction cosines are changed. Hence the direction cosines of an undirected line are *ambiguous in sign.*

In Fig. 78, we have

$$x^2 + y^2 + z^2 = \rho^2,$$

or

$$l^2\rho^2 + m^2\rho^2 + n^2\rho^2 = \rho^2.$$

This gives the very important result:

The direction cosines of any line satisfy the relation

$$l^2 + m^2 + n^2 = 1. \tag{1}$$

It should also be pointed out that, given any set of numbers l, m, n satisfying (1), there will always be a line having those numbers as direction cosines; in fact, the line in question is the line through (0, 0, 0), (l, m, n).

86. Direction components. If the direction cosines of a line are proportional to three numbers a, b, c, the actual values of the cosines must be

$$l = ka, \qquad m = kb, \qquad n = kc,$$

where k is a quantity as yet undetermined. Substituting these values of l, m, and n in (1), § 85, we get

$$k^2(a^2 + b^2 + c^2) = 1, \text{ or } k = \frac{1}{\sqrt{a^2 + b^2 + c^2}}.$$

Employing this value of k in the equations above, we find:

If the direction cosines of a line are proportional to any three numbers a, b, c, their actual values are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \qquad m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \tag{1}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

Any set of numbers a, b, c proportional to the direction cosines of a line are called *direction components* for that line.

The ambiguity of sign mentioned in § 85 appears here in the fact that we might equally well choose the negative sign before the radical.

87. Radius vector of a point. The directed segment OP from the origin to the point $P : (x, y, z)$ is called the *radius vector* of P; its length is

$$\rho = \sqrt{x^2 + y^2 + z^2}. \tag{1}$$

By § 85, the coördinates of the point in terms of the radius vector and its direction cosines are

(2) $$x = l\rho, \qquad y = m\rho, \qquad z = n\rho.$$

Hence *the direction cosines of the radius vector of a point are proportional to the coördinates of the point.*

88. Direction cosines of the line through two points. Let d be the distance between the points $P_1 : (x_1, y_1, z_1)$, $P_2 : (x_2, y_2, z_2)$. Then the direction cosines of the line P_1P_2 are

$$l = \frac{P_1L}{d} = \frac{x_2 - x_1}{d},$$

$$m = \frac{P_1M}{d} = \frac{y_2 - y_1}{d},$$

$$n = \frac{P_1N}{d} = \frac{z_2 - z_1}{d}.$$

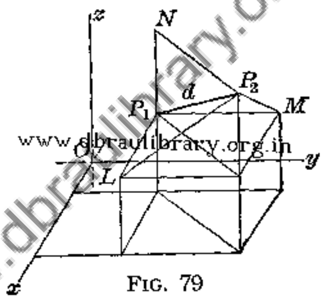

FIG. 79

Hence:

The direction cosines of the line joining the points (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *are proportional to* $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.

EXERCISES

1. A line has the direction cosines $l = \frac{3}{10}$, $m = \frac{2}{5}$. What angle does it make with Oz? *Ans.* 30°.

2. A line has direction components 3, 4, 5. What angle does it make with Oz? *Ans.* 45°.

Draw the following lines.

3. Through O, with direction components 3, 1, 2.

4. Through (3, 5, 1), with direction components 2, 3, − 1.

5. For each of the following points, find the length and direction cosines of the radius vector: (*a*) (2, 3, 3); (*b*) (− 3, 5, − 2); (*c*) (2, 1, 0).

6. Where must a point lie if its radius vector has (*a*) $l = 0$? (*b*) $l = m = 0$? (*c*) $l = \frac{1}{2}$? (*d*) $l = 1$? (*e*) $l = n = \frac{1}{2}\sqrt{2}$?

7. A point is at the distance 5 from O, and its radius vector makes an angle of 60° with Oy and 45° with Oz. Find its coördinates.

8. A point is at the distance 3 from O, and its radius vector has $l = \frac{1}{3}$, $m = \frac{5}{6}$. Find its coördinates. *Ans.* $(1, \frac{5}{2}, \pm \frac{1}{2}\sqrt{7})$.

9. A point is at the distance $2\sqrt{5}$ from Oy, 8 from O, and its radius vector makes an angle $\frac{1}{3}\pi$ with Ox. Find its coördinates.

10. A point is at the distance $2\sqrt{3}$ from Oz, 4 from O, and its radius vector has $l = \frac{3}{4}$. Find its coördinates. *Ans.* $(3, \pm\sqrt{3}, \pm 2)$.

11. A point is at the distance 4 from the yz-plane, and its radius vector has $m = \frac{3}{4}$, $n = \frac{1}{4}\sqrt{6}$. Find the point. *Ans.* $(4, 12, 4\sqrt{6})$.

12. A point is at distance 5 from O, 4 from the yz-plane, and its radius vector has $n = \frac{1}{5}$. Find the point. *Ans.* $(4, \pm 2\sqrt{2}, 1)$.

13. A point is distant 1 from the yz-plane, 2 from the zx-plane, and its radius vector has $n = \frac{1}{3}$. Find the point. *Ans.* $(1, 2, \frac{1}{4}\sqrt{10})$.

14. A point is distant $4\sqrt{10}$ from Ox, and its radius vector has direction components 3, 2, 6. Find the point. *Ans.* $(\pm 6, \pm 4, \pm 12)$.

15. A point is at the distance $\sqrt{5}$ from the z-axis, 1 from the xy-plane, and its radius vector makes an angle $\frac{1}{4}\pi$ with Ox. Find its coördinates.

16. A point is at the distance $\sqrt{2}$ from Oz, $\sqrt{3}$ from Ox, and its radius vector makes an angle of 45° with Oz. Find its coördinates.

17. A point is at the distance $2\sqrt{10}$ from Oz, $2\sqrt{7}$ from Oy, and its radius vector has $l = \frac{1}{4}$. Find the point. *Ans.* $(2, \pm 6, \pm 2\sqrt{6})$.

18. Find the direction cosines of the sides of the triangle having the vertices $(4, 1, -2)$, $(6, 0, 4)$, $(-2, 3, -4)$.

19. Prove that the points $(6, 1, -3)$, $(0, -2, 3)$, $(10, 3, -7)$ lie in a straight line.

20. Prove that the points $(1, -1, 3)$, $(3, 2, 1)$, $(-3, -7, 7)$ lie in a straight line.

21. Prove that the points $(3, 3, 3)$, $(1, 2, -1)$, $(4, 1, 1)$, $(6, 2, 5)$ are the vertices of a parallelogram.

89. Projections. The *projection* of a point P upon any line is defined as the foot of the perpendicular from P to that line. The projection of a line segment P_1P_2 upon any line is the segment joining the projections of the endpoints P_1, P_2 upon that line.

The projection of a broken line upon any line is the sum of the projections of the segments forming the broken line. *The projection of a broken line $P_1P_2 \cdots P_n$ upon any line is equal to the projection of the closing line P_1P_n upon that line.*

Thus, in Fig. 80,

$$L_1L_2 + L_2L_3 + L_3L_4 = L_1L_4.$$

It should be noted that the segments composing the broken line need not all lie in the same plane.

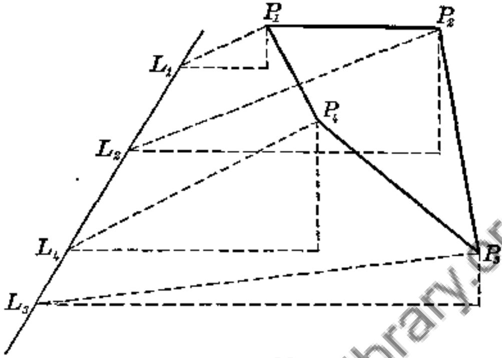

FIG. 80

90. Angle between two lines. To find the angle ϕ between any two lines λ_1, λ_2 intersecting at the origin, let us denote the direction cosines of λ_1 by l_1, m_1, n_1, those of λ_2 by l_2, m_2, n_2, and choose on λ_1 any point $P:(x, y, z)$ with radius vector ρ. Then

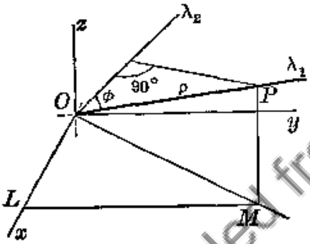

FIG. 81

$$OL = x = l_1\rho,$$
$$LM = y = m_1\rho,$$
$$MP = z = n_1\rho.$$

Now the projection of the broken line $OLMP$ on λ_2 equals the projection of OP on λ_2:

$$OP \cos\phi = OL \cdot l_2 + LM \cdot m_2 + MP \cdot n_2,$$

or, by the above expressions for OL, LM, MP,

$$\rho \cos\phi = l_1\rho l_2 + m_1\rho m_2 + n_1\rho n_2,$$

(1) $$\cos\phi = l_1l_2 + m_1m_2 + n_1n_2.$$

If the two lines have direction components a_1, b_1, c_1 and a_2, b_2, c_2, the direction cosines may be found by § 86. Substituting in formula (1), we find

(2) $$\cos\phi = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Of course these results apply at once to any two intersecting lines. Further, the angle between two non-intersecting lines is defined as being *equal to the angle between two intersecting lines that are respectively parallel to the given lines.* With this convention the formulas give the angle between any two lines in space.

Example: Find the angle between the lines joining the points (3, 1, 2), (4, 0, 4) and (− 2, 4, 4), (0, − 1, 3).

By § 88, the direction components of the lines are respectively 1, − 1, 2 and 2, − 5, − 1. By (2), we find

$$\cos\phi = \frac{2 + 5 - 2}{\sqrt{6}\cdot\sqrt{30}} = \frac{1}{6}\sqrt{5}.$$

91. Perpendicular lines. In formula (1), § 90, if $\cos\phi = 0$, then $\phi = 90°$, and we have the

THEOREM: *Two lines having the direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are perpendicular if and only if*

(1) $$l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

COROLLARY: *Two lines with direction components a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular if and only if*

(2) $$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

EXERCISES

1. Find the angle between two lines whose direction components are 2, 1, − 2 and 1, 1, 0. *Ans.* 45°.

2. Find the angle between two lines whose direction components are 1, 2, 3 and 2, 2, − 1. *Ans.* $\cos\phi = \frac{1}{14}\sqrt{14}$.

3. Find the angle between the radius vectors of the points (3, − 4, 5) and (− 1, − 1, 0). *Ans.* $\cos\phi = \frac{1}{10}$.

4. Find the angle between the radius vectors of the points (2, − 2, 2) and (2, 1, 2). *Ans.* $\cos\phi = \frac{1}{3}\sqrt{3}$.

5. Find the angle between the lines joining the points (3, 1 − 2), (1, − 2, 4) and (− 4, 8, 0), (5, 2, − 2). *Ans.* $\cos\phi = \frac{13}{77}$.

6. Find the angle between the lines joining the points (3, 1, − 2), (4, 0, − 4) and (4, − 3, 3), (6, − 2, 2). *Ans.* $\frac{1}{3}\pi$.

Solve the following by a new method.

7. Ex. 15(*a*), p. 126.

8. Ex. 16(*a*), p. 126.

9. Ex. 17(*a*), p. 126.

10. Ex. 18(*a*), p. 126.

11. Ex. 19, p. 126.

12. Ex. 21, p. 130.

13. Prove that the points (1, 3, − 3), (2, 2, − 1), (3, 4, − 2) form an equilateral triangle.

14. Solve Ex. 13 by another method.

CHAPTER XIV

THE PLANE

92. The locus of an equation. If x, y, and z are connected by an equation of any form, we may assign values at pleasure to two of the variables and compute the third, thus determining certain sets of values of x, y, and z satisfying the equation. Each of these sets of values may be considered as the Cartesian coördinates of a point. The points whose coördinates satisfy the equation are not scattered at random throughout space; instead, they form a definite *surface*, called the *locus* of the equation:

In space of three dimensions, the locus of an equation is a surface* containing those points, and only those points, whose coördinates satisfy the equation.

Examples of the correspondence between an algebraic equation and a surface in space are furnished by Exs. 21–22, p. 126. Of course the situation is entirely analogous to the one arising in Chap. III.

93. Equations in one variable. The equation

$$x = k$$

represents a plane parallel to the yz-plane at a distance k from it; for that plane contains all those points, and only those points, whose x-coördinate is k. An analogous result holds for an equation of the same form in y or in z. Hence:

An equation of the first degree in one variable represents a plane parallel to the plane of the other two variables; and conversely.

* Exceptionally, a surface may reduce to a single line or a single point, etc.; or there may be no locus at all. Such forms are unimportant in a first course.

94. Plane sections of a surface; traces. A plane and a surface intersect in general in a curve, called the *section* of the surface by the plane. Of particular importance are the sections of a surface by the coördinate planes; these sections will be called for brevity the *traces* of the surface.

If in the equation of the surface we substitute $z = k$, the resulting equation in x and y, considered as the equation of a curve in the plane $z = k$, represents the section of the surface by that plane. In particular, *to obtain the xy-trace, we set* $z = 0$. Similarly for the other traces.

95. Normal form of the equation of a plane. Let $P : (x, y, z)$ be any point of the plane RST, and N the foot of the perpendicular from O upon the plane. Denote the length of the normal ON by p and its direction cosines by l, m, n. By (2), § 87, the coördinates of N are (lp, mp, np), so that (§ 88) the direction components of NP are $x - lp$, $y - mp$, $z - np$. Hence (§ 91) the lines are perpendicular if and only if

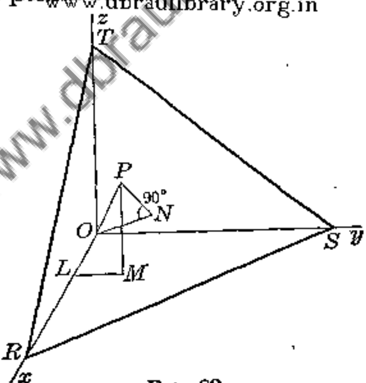

FIG. 82

$$l(x - lp) + m(y - mp) + n(z - np) = 0,$$
$$lx + my + nz = l^2p + m^2p + n^2p,$$

or, by (1), § 85,

$$lx + my + nz = p. \qquad (1)$$

This is the *normal form* of the equation of the plane. For definiteness, the convention is adopted that p shall be always positive.

The proof is easily modified to cover the case in which the plane passes through the origin, and the same equation is obtained. Hence, we have proved the

THEOREM: *The equation of a plane is always of the first degree.**

96. General form; reduction to normal form. Every equation of the first degree can be written in the form

(1) $$Ax + By + Cz + D = 0.$$

Let us transpose the constant term to the right member and make it positive (by changing all signs if necessary), and then divide through by $\sqrt{A^2 + B^2 + C^2}$:

(2) $$\pm \frac{A}{\sqrt{A^2 + B^2 + C^2}} x \pm \frac{B}{\sqrt{A^2 + B^2 + C^2}} y \pm \frac{C}{\sqrt{A^2 + B^2 + C^2}} z = \frac{\mp D}{\sqrt{A^2 + B^2 + C^2}}.$$

Now the coefficients of x, y, and z are the direction cosines of a certain line, since the sum of their squares is unity: hence (2) is *the equation of a plane in the normal form.*

From the fact that this reduction can always be carried out (since $\sqrt{A^2 + B^2 + C^2}$ cannot be 0), we deduce the

THEOREM: *Every equation of the first degree represents a plane.*

Further, we have the

RULE: *To reduce the equation of any plane*

$$Ax + By + Cz + D = 0$$

to the normal form, divide by $\sqrt{A^2 + B^2 + C^2}$ *and choose signs so that the constant is positive in the right member.*

97. Perpendicular line and plane. From the fact that the coefficients A, B, C in the general equation of the plane are proportional to the direction cosines of the normal, we obtain at once the very important

* See the footnote, p. 36.

THEOREM: *If a line and plane are perpendicular, the coefficients of x, y, z in the equation of the plane may be taken as direction components of the line, and vice versa.*

Example: Write the equation of a plane perpendicular to the radius vector of the point (3, 1, 2) and passing through (0, 4, 5).

By § 87, the direction components of the radius vector are 3, 1, 2; by the above theorem, these numbers may be taken as the coefficients of x, y, z in the equation of the plane. The result is

$$3x + y + 2z = 14,$$

the right member being necessarily the value assumed by the left member when the coördinates (0, 4, 5) are substituted (cf. the example of § 27).

98. Equations in two variables. Consider an equation of first degree from which z is missing — for definiteness take the equation

(1) $\qquad x + 2y = 4.$

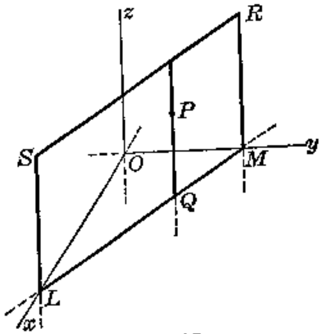

FIG. 83

The xy-trace of this surface is the line LM. Let Q be any point of that line, and P any point vertically above or below Q. Since the coördinates of Q satisfy (1), and the coördinates of P are the same as those of Q except for a different z, which is not involved, it follows that P, and hence the entire vertical line through Q, lies in the locus. Since Q is any point of LM, the locus consists of all straight lines perpendicular to the xy-plane through points of the line LM, and is therefore a plane perpendicular to the xy-plane.

Evidently a similar result holds for any equation of first degree in which x, y, or z is missing. Hence the

THEOREM: *An equation of the first degree in two variables represents a plane perpendicular to the plane of those two variables; and conversely.*

EXERCISES

For the given plane, find the intercepts on the axes; write the equations of the traces and draw the traces; reduce the equation to the normal form and read off the direction cosines of the normal and the distance from the origin.

1. $x + 3y + 4z = 6.$

2. $3x + 3y + 5z = 15.$

3. $x + y - 2z = 4.$

4. $2x - y - z = 2.$

5. $x - 3y + 4z = 0.$

6. $2x - y - 2z = 0.$

7. $x + 2z = 3.$

8. $2y + 3z = 24.$

Find the equations of the following planes.

9. At a distance 3 from O, with the normal having direction components 2, -1, -2. *Ans.* $2x - y - 2z = \pm 9.$

10. Through $(3, 1, -1)$ perpendicular to the radius vector of that point. *Ans.* $3x + y - z = 11.$

11. Through $(-2, 1, -3)$ perpendicular to the radius vector of that point. *Ans.* $2x - y + 3z + 14 = 0.$

12. With the point $(-2, 4, 2)$ as the foot of the normal from O.

13. Through $(-3, 1, 4)$ perpendicular to the radius vector of the point $(1, -2, 2)$. *Ans.* $x - 2y + 2z = 3.$

14. Through $(0, 1, 0)$ perpendicular to the line joining $(2, -3, 1)$, $(3, 4, 0)$. *Ans.* $x + 7y - z = 7.$

15. Perpendicular to the radius vector of $(1, 3, -2)$ and passing at a distance 2 from the origin. *Ans.* $x + 3y - 2z = \pm 2\sqrt{14}.$

16. Perpendicular to the radius vector of $(3, -2, 2)$ and passing at a distance 3 from the origin. *Ans.* $3x - 2y + 2z = \pm 3\sqrt{17}.$

17. The base of an isosceles triangle joins the points $(3, 1, -2)$, $(-1, 3, 0)$. Find the locus of the third vertex by § 97.

18. Solve Ex. 17 by another method (§ 84).

19. One side of a right triangle joins the points $(1, 5, -1)$, $(-2, 3, 0)$, with the right angle at the latter point. Find the locus of the third vertex.

20. Derive the normal form from the fact that, in Fig. 82, the projection of the broken line $OLMPN$ upon the normal must equal ON.

21. Derive the normal form by applying the Theorem of Pythagoras to the triangle ONP. Does the derivation hold in all cases?

22. By reducing the equations to the normal form, show that:
If the equations of two planes differ only in the constant term, the planes are parallel; conversely, if two planes are parallel, their equations can be made to differ only in the constant term.

Using the theorem of Ex. 22, find the equation of the plane.

23. Through $(-2, 3, 4)$ parallel to the plane $2x - y - 2z = 5$.

24. Through $(3, 4, -2)$ parallel to the plane $x - y + 3z = 4$.

25. Through $(5, 0, 2)$ parallel to the plane $z = 2x + 3y$.

26. Through $(1, 5, -2)$ parallel to the plane $y = 3x + 4z + 3$.

27. Parallel to the plane $6x + 3y - 2z = 14$ and (*a*) half as far from O; (*b*) at the distance 3 from O. *Ans.* (*a*) $6x + 3y - 2z = \pm 7$.

28. Parallel to the plane $x - 4y - 8z + 27 = 0$ and (*a*) 3 units farther from the origin; (*b*) at the distance 1 from the given plane; (*c*) 2 units nearer the origin. *Ans.* (*c*) $x - 4y - 8z = \pm 9$.

29. Parallel to the plane $x - y - z + 5 = 0$ and passing at the distance $2\sqrt{3}$ from $(1, 2, -2)$. *Ans.* $x - y - z = 1 \pm 6$.

30. Parallel to the plane $2x - 3y - 5z + 1 = 0$ and passing at the distance 3 from $(-1, 3, 1)$.

31. Parallel to the plane $2x + 2y + z = 0$ and passing (*a*) half as far from $(1, 3, -2)$; (*b*) 2 units farther from $(1, 3, -2)$; (*c*) at a distance 3 from $(1, 3, -2)$. *Ans.* (*b*) $2x + 2y + z = 6 \pm 12$.

32. Parallel to the plane $2x - 6y + 9z = 0$ and (*a*) twice as far from $(5, 1, 2)$; (*b*) 1 unit farther from $(5, 1, 2)$; (*c*) at a distance 3 from $(5, 1, 2)$.

Find the distance between the given planes.

33. $2x - y + 3z + 5 = 0,\quad 2x - y + 3z + 2 = 0$.

34. $3x - y - z = 6,\quad 6x - 2y - 2z = 5$.

35. $x + 3y + 3z = 4,\quad 3x + 9y + 9z + 8 = 0$.

36. $2x - 5y + z + 1 = 0,\quad 2x - 5y + z = 3$.

37. Two faces of a cube lie in the planes $x + 2y + 2z - 3 = 0$, $3x + 6y + 6z + 2 = 0$. Find the volume of the cube.

38. Show that the planes $x + y - z + 1 = 0$, $x + y - z - 1 = 0$, $3x - 2y + z + 3 = 0$, $3x - 2y + z + 2 = 0$, $x + 4y + 5z + 20 = 0$, $x + 4y + 5z = 1$ form a box, and find its volume. *Ans.* $V = 1$.

99. Plane through a given point. If the plane

(1) $$Ax + By + Cz + D = 0$$

is to pass through the point (x_1, y_1, z_1), those coördinates must satisfy the equation, and we have

$$Ax_1 + By_1 + Cz_1 + D = 0.$$

By subtraction, we find that *the equation of any plane through the point* (x_1, y_1, z_1) *may be written in the form*

(2) $$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.$$

100. Plane determined by three points. From the fact that equation (1), § 99, contains three essential constants — viz. the ratios of any three of the quantities A, B, C, D to the fourth — we conclude that *a plane is determined by three points,* or by any similar set of three conditions.

To find the equation of the plane through three points, the obvious method would be to substitute the coördinates of the points in turn in (1), § 99. The solution may, however, be expedited by use of (2), § 99, as in the following

Example: Find the equation of the plane through the points (2, 4, 3), (1, 3, 1), (− 1, − 1, − 4).

The equation of any plane through (2, 4, 3) is

(1) $$A(x - 2) + B(y - 4) + C(z - 3) = 0.$$

Substitute the coördinates of the other points in (1):

$$-A - B - 2C = 0,$$
$$-3A - 5B - 7C = 0.$$

Solving for A and C in terms of B, we find

$$A = 3B, \qquad C = -2B.$$

By substitution in (1) the equation of the plane is found to be

$$3B(x - 2) + B(y - 4) - 2B(z - 3) = 0,$$

or

$$3x + y - 2z = 4.$$

101. Perpendicular planes. Two planes

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0$$

are perpendicular if their normals are perpendicular to each other. The direction cosines of the normals are proportional to A_1, B_1, C_1 and A_2, B_2, C_2 respectively. Applying the condition of perpendicularity (Corollary, § 91) to these two lines, we obtain the following

THEOREM: *Two planes*

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0$$

are perpendicular if

(1) $$\mathbf{A_1A_2 + B_1B_2 + C_1C_2 = 0;}$$

and conversely.

Example: Find the equation of a plane through the points (1, 1, 2), (2, 4, 3), and perpendicular to the plane

(2) $$x - 3y + 7z + 5 = 0.$$

The equation of any plane through (1, 1, 2) is

(3) $$A(x - 1) + B(y - 1) + C(z - 2) = 0.$$

Substituting the coördinates (2, 4, 3) in (3), we get

(4) $$A + 3B + C = 0.$$

Condition (1) applied to the planes (2) and (3) gives

(5) $$A - 3B + 7C = 0.$$

From (4) and (5) we find

$$A = -4C, B = C,$$

whence the equation of the plane is

$$-4C(x - 1) + C(y - 1) + C(z - 2) = 0,$$

or

$$4x - y - z = 1.$$

EXERCISES

Find the equation of the plane through the given points.

1. (2, 3, − 3), (1, 1, − 2), (− 1, 1, 4). *Ans.* $3x - y + z = 0$.

2. (2, 2, 2), (3, 1, 1), (6, − 4, − 6). *Ans.* $x + 2y - z = 4$.

3. (3, 3, 1), (− 3, 2, − 1), (8, 6, 3). *Ans.* $4x + 2y - 13z = 5$.

4. (− 1, 1, − 5), (3, − 1, 5), (− 2, 3, 0). *Ans.* $5x + 5y - z = 5$.

Find the equations of the following planes.

5. Through the points (1, 2, 3), (3, 2, − 1), perpendicular to the plane $3x + 2y + 6z + 4 = 0$. *Ans.* $2x - 6y + z + 7 = 0$.

6. Through the points (0, 2, 3), (5, − 1, 4), perpendicular to the plane $x + 2y + 3 = 0$. *Ans.* $2x - y - 13z + 41 = 0$.

7. Through the points (2, 1, 1), (3, 2, 2), perpendicular to the plane $x + 2y - 5z = 3$. *Ans.* $7x - 6y - z = 7$.

8. Through (4, 1, 0), perpendicular to the planes $2x - y - 4z = 6$, $x + y + 2z = 3$. *Ans.* $2x - 8y + 3z = 0$.

9. Through (1, 1, 2), perpendicular to the planes $2x - 2y - 4z = 3$, $3x + y + 6z = 4$. *Ans.* $x + 3y - z = 2$.

10. Perpendicular to the planes $y = 3x + z$, $x + 5y + 3z = 0$, and passing at a distance $\sqrt{6}$ from the origin. *Ans.* $x + y - 2z = \pm 6$.

11. Perpendicular to the planes $z = 4y - x$, $3x + 4y + z = 2$, and passing at a distance 1 from the origin. *Ans.* $4x - y - 8z = \pm 9$.

12. Perpendicular to the planes $2x - y + z = 1$, $3x + y = 6z$, and passing at a distance 2 from (1, 2, − 1). *Ans.* $x + 3y + z = 6 \pm 2\sqrt{11}$.

13. Prove that the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

represents the plane determined by the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

14. In Ex. 13, what happens if the minors of all the elements in the first row are 0? Explain geometrically.

15. Use the formula of Ex. 13 to solve Exs. 1–4.

16. Find the equation of the plane whose intercepts on the x-, y-, and z-axes are respectively a, b, c (the *intercept form*).

Ans. $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

17. Prove the theorem of § 98 by a new method (§ 101).

CHAPTER XV

THE STRAIGHT LINE

102. Representation of a curve in space. Two surfaces intersect in general in a curve. If the equations of the two surfaces be considered as *simultaneous*, their locus consists of all points lying on *both* surfaces. Hence:

The locus of two simultaneous equations is a curve, viz. the curve of intersection of the surfaces represented by the two equations separately.

103. The straight line. Two equations of first degree represent, of course, two planes. When considered as *simultaneous*, the equations represent all points common to the two planes. From the fact that two planes (if not parallel) intersect in a straight line, we deduce the following

THEOREM: *The locus of two simultaneous equations of the first degree is a straight line.*

Thus the *general form* of the equations of a straight line is

$$(1) \qquad \begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases}$$

Since through any line an infinite number of planes may be passed, it is obvious that a straight line may be represented by two simultaneous equations in an infinite number of ways — viz. by the equations of any two planes through the line. Of these various equations there are usually one or two pairs especially simple and convenient (see §§ 105, 106).

104. Planes through a given line. Let the equations of a straight line be denoted by

(1) $$u = 0, \qquad v = 0$$

where u and v represent any expressions of the first degree in x, y, and z. Then the equation

(2) $$u + kv = 0,$$

where k is a constant, is an equation of the first degree and thus represents a plane; further, since (2) is satisfied whenever both the equations (1) are satisfied, the plane (2) contains all points of the line (1). (Cf. § 37.) Hence the

THEOREM: *If*

$$u = 0, \qquad v = 0$$

are the equations of any line, the equation

(3) $$u + kv = 0,$$

where k is constant, represents a plane through the given line.

Any two equations of the form (3) may of course be used to represent the line, instead of the original pair.

105. Projecting planes. The planes through a line perpendicular to the coördinate planes are called the *projecting planes*, and their traces are the *projections*, of the line on the coördinate planes. It is often convenient, particularly in making a sketch, to represent a line by the equations of two of its projecting planes.

If in (3), § 104, we choose k so that the term in x drops out, which amounts merely to eliminating x between the equations of the line, the resulting equation represents a plane through the given line perpendicular to the yz-plane (§ 98). In this way we establish the

RULE: *To obtain the yz-projecting plane of a line, eliminate x between the equations of the line; similarly for the other projecting planes.*

Example: Draw the line

$$4x + 4y + 3z = 14,$$
$$x - 2y + 3z = 2.$$

Eliminating z and y in turn, we find the xy- and zx-projecting planes:

$$x + 2y = 4,$$
$$2x + 3z = 6.$$

The xy-traces of these planes are the lines LQ, MP_1; the yz-traces are LP_2, NP_2. The points P_1, P_2 are the xy- and yz-*piercing points* of the given line.

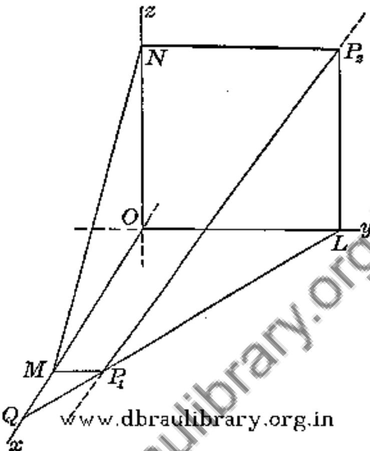

FIG. 84

EXERCISES

Find the equations of the following planes.

1. Through the line $x + y = 2$, $z = 4$ and the point (2, 4, 0). Draw the figure. *Ans.* $x + y + z = 6$.

2. Through the line $x + 2z = 2$, $y = 2$ and the point (4, 1, 0). Draw the figure. *Ans.* $x + 2y + 2z = 6$.

3. Through the line $3x + y - z = 3$, $x + 2y + 4z + 4 = 0$ and perpendicular to the plane $x - 2y - z = 5$. *Ans.* $23x + 11y + z = 13$.

4. Through the line $x - y = 3$, $y - 2z = 4$ and perpendicular to the plane $3x - 5y + 6z = 2$. *Ans.* $17x - 9y - 16z = 83$.

5. Show that the line $2x + y - 3z = 4$, $3x - y + 2z = 2$ lies in the plane $4x + 7y - 19z = 16$.

6. Show that the line $2x + 3y - z = 4$, $3x - y - 2z = 1$ lies in the plane $x - 15y - 2z + 13 = 0$.

Find the projecting planes of the following lines, and draw the lines.

7. $x + y + 3z = 3$, $2x + y + 4z = 4$.

8. $4x + y + z = 4$, $3x + 3y + 2z = 6$.

9. $4x + 3y + z = 18$, $2x - 3y + 2z = 0$.

10. $4x + y + 3z = 9$, $5x + y + 6z = 15$.

11. $2x + y - z = 3$, $3x + 2y - z = 6$.

12. $3x - 2y - z = 0$, $x - y + 2z = 0$.

106. Symmetric form. The equations of the line through (x_1, y_1, z_1) having direction components a, b, c may be found as follows. If (x, y, z) is any point of the line, the direction cosines are also proportional to $x - x_1$, $y - y_1$, $z - z_1$. Since two sets of numbers proportional to the same third set are proportional to each other, we have

$$(1) \qquad \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Formula (1) is called the *symmetric form;* for most purposes it is more convenient than any other.

Example: Write the equations of the line joining the points $(2, 3, 1)$, $(1, -3, -1)$.

By § 88, the direction components are 1, 6, 2; hence the equations of the line are

$$(2) \qquad \frac{x - 2}{1} = \frac{y - 3}{6} = \frac{z - 1}{2}.$$

107. Determination of direction cosines; reduction to the symmetric form. To reduce the equations of a line to the symmetric form, we may proceed, in general, as in the

Example: Reduce the equations

$$4x + 4y + 3z = 14, \qquad x - 2y + 3z = 2$$

to the symmetric form (cf. the example of § 105).

Find any two of the projecting planes: for instance,

$$x + 2y = 4, \qquad 2x + 3z = 6.$$

These equations contain one variable in common; *solve for that variable and equate values:*

$$x = -2y + 4 = \frac{-3z + 6}{2}.$$

Divide through by such a number as to reduce the coefficient of each variable to unity — in this case by -6:

$$\frac{x}{-6} = \frac{y - 2}{3} = \frac{z - 2}{4}.$$

If the line is parallel to a coördinate plane, the above process fails, since the equations of the projecting planes will not contain one variable in common. In that case, we may find two points of the line and then employ § 88.

EXERCISES

Write the equations of the line through the given points.

1. $(4, 2, 1), (-1, 3, 6)$. **2.** $(3, -1, 4), (4, -2, -3)$.

3. $(8, -5, 6), (0, 1, 0)$. **4.** $(0, -5, -2), (1, 3, 3)$.

Reduce the following equations to the symmetric form.

5. $x + 2y + 4z + 1 = 0, \quad x - 5y - 5z = 2$.

6. $x + y - z = 0, \quad 2x - 2y - z = 2$.

7. Show that the lines $x + y - z + 4 = 0$, $3x + z = 1$ and $3x - y - z = 0$, $4x - 6y + z = 2$ are parallel.

8. Show that the lines $7x + 5y - 7z = 3$, $9x + 3y - z = 0$ and $2x + y - z = 1$, $x - y + 3z + 3 = 0$ are parallel.

9. Find the equation of the plane determined by the parallel lines of Ex. 7. *Ans.* $x - 5y + 2z = 2$.

10. Find the equation of the plane determined by the parallel lines of Ex. 8. *Ans.* $29x + 7y + 3z + 3 = 0$.

11. Find the point of intersection of the intersecting lines $y = 2x$, $2x + y + 2z = 2$ and $2x - y + 2z + 2 = 0$, $3x + y + z = 4$.

12. Find the point of intersection of the intersecting lines $x + y = z$, $2x - y - 3z = 1$ and $x + 2y + z = 3$, $3x + 3y - 2z = 2$.

13. Find the equation of the plane determined by the lines of Ex. 11. *Ans.* $18x + y + 10z = 10$.

14. Find the equation of the plane through the lines of Ex. 12. *Ans.* $12x + 9y - 13z = 1$.

15. Find the distance from the point $(3, 0, 2)$ to the line $2x - y - z = 1$, $x = y$. (Pass a plane through the point and the line, then a plane through the line perpendicular to the plane just determined.) *Ans.* $\sqrt{6}$.

16. Find the distance from the point $(2, 3, 1)$ to the line $y + z = 1$, $2x - 3y - 2z + 4 = 0$. (Note the suggestion in Ex. 15.) *Ans.* 3.

CHAPTER XVI

SURFACES

108. Symmetry. It follows from §§ 92, 95 that an equation not of the first degree represents in general a *curved surface* of some kind. We begin our study of surfaces with a brief discussion of symmetry.

Two points P_1, P_2 are said to be *symmetric with respect to a plane* if the plane is perpendicular to the line P_1P_2 at its midpoint, i.e., if P_2 is the image, or reflection, of P_1 in that plane. A geometric figure is symmetric with respect to a plane if corresponding to every point P_1 of the figure the image P_2 also belongs to the figure.

The most important type of symmetry of space figures is that with respect to the coördinate planes. We have at once the

THEOREM: *A surface is symmetric with respect to the yz-plane if x can be replaced by $-x$ without changing the equation; and conversely.* Similarly for symmetry with respect to the *zx*- and *xy*-planes.

109. Surfaces of revolution. A *surface of revolution* is a surface that can be generated by rotating a curve about a straight line. The straight line is called the *axis of revolution*, and the revolved curve is the *generating curve*, or *generator*. The sections by planes through the axis are *meridians;* those by planes perpendicular to the axis are *right sections*, or *parallels*. Evidently the right sections are circles with centers in the axis of revolution.

The more important surfaces of revolution are as follows.

Surface	*Generated by the rotation of*
Sphere	A circle about a diameter.
Prolate spheroid	An ellipse about its major axis.
Oblate spheroid	An ellipse about its minor axis.
Hyperboloid of revolution of one sheet	A hyperbola about its conjugate axis.
Hyperboloid of revolution of two sheets	A hyperbola about its transverse axis.
Paraboloid of revolution	A parabola about its axis.
Circular cylinder	A straight line about a line parallel to it.
Circular cone	A straight line about a line intersecting it obliquely.
Torus	A circle about a line in its plane, not intersecting it.

The spheroids are also called *ellipsoids of revolution.* The equations of all of these surfaces, with the exception of the torus, are of the second degree.

Example: Sketch the surface

$$x^2 + y^2 = az \qquad (a > 0).$$

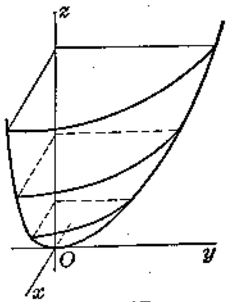

FIG. 85

The surface is symmetric with respect to the yz- and zx-planes (§ 108). Setting x, y, z in turn equal to 0, we get as traces the parabola $y^2 = az$; the parabola $x^2 = az$; the point-circle $x^2 + y^2 = 0$. Setting $z = k$ (i.e. taking sections parallel to the xy-plane), we get for $k < 0$, no curve; for $k > 0$, the circles $x^2 + y^2 = ak$. This shows that the surface is a *paraboloid of revolution* generated by rotating either of the above parabolas about Oz. On account of the symmetry, we need to make only a first-octant sketch.

110. The sphere. A *sphere* is the locus of a moving point that remains at a constant distance, the *radius*, from a fixed point, the *center*.

By (1), § 84, the equation of a sphere of radius a with center at the point (h, k, l) is

(1) $$(x - h)^2 + (y - k)^2 + (z - l)^2 = a^2.$$

Every equation of the form

(2) $$Ax^2 + Ay^2 + Az^2 + Gx + Hy + Iz + K = 0$$

can be reduced to the form (1) by completing the squares in x, y, and z; hence *every equation of the form* (2) *represents a sphere* (exceptionally, a point, or no locus — cf. § 34).

EXERCISES

By a study of the sections, show that the following equations represent surfaces of revolution, locate the axis, and determine a generating curve. Classify the surfaces and draw the figures.

1. $4x^2 + y^2 + z^2 = 4$.
2. $2x^2 + y^2 + 2z^2 = 4$.
3. $x^2 + y^2 = 4a^2$.
4. $y^2 + z^2 = a^2$.
5. $x^2 + z^2 = 4y$.
6. $x^2 + y^2 - z^2 = 9$.
7. $4x^2 + 4y^2 = z^2$.
8. $x^2 - y^2 + z^2 = 0$.
9. $4x^2 - 9y^2 + 4z^2 + 36 = 0$.
10. $4y^2 + 4z^2 = x$.
11. $4x^2 - y^2 - z^2 + 4 = 0$.
12. $x^2 - y^2 - z^2 = 1$.
13. $x^2 + y^2 = 2ax$.
14. $x^2 + y^2 + 9z^2 = 36$.
15. $25x^2 + 25y^2 + z^2 = 25$.
16. $x^2 - 9y^2 - 9z^2 = 0$.
17. $y^2 + z^2 + x = 4$.
18. $y^2 + z^2 - 2y = 0$.

Find the center and radius of each of the following spheres.

19. $x^2 + y^2 + z^2 - 4x + 2y + 6z - 2 = 0$.
20. $x^2 + y^2 + z^2 + 3x - y - 2z = 2$.
21. $x^2 + y^2 + z^2 = 4x - 2y$.
22. $4x^2 + 4y^2 + 4z^2 = z$.

Write the equations of the following spheres.

23. Of radius 2 with center at $(1, -3, -5)$.
24. With center at O tangent to the plane $9x - 2y + 6z + 11 = 0$.
25. With center at $(2, 1, -3)$ tangent to the plane $x + y - z = 3$.

26. With center at (0, 3, − 5) tangent to the plane $2x - 3y - z = 3$.

27. Prove that a sphere is determined by four points.

28. Find the equation of the sphere through the points (0, 0, 0), (− 1, 0, 0), (0, 4, 0), (1, 2, − 1). *Ans.* $x^2 + y^2 + z^2 + x - 4y - z = 0$.

111. Quadric surfaces. A surface whose equation is of the second degree is called a *quadric surface*. Next to the plane, the quadrics form by far the most important class of surfaces.

The quadrics proper are of nine species: the ellipsoid (of which the sphere is a special case), the hyperboloids (two species), the paraboloids (two species), the quadric cylinder (three species), and the quadric cone. In addition there are the degenerate forms, analogous to those of Chap. VI — two parallel, coincident, or intersecting planes, a single straight line, or a point.

112. The ellipsoid. The surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

is called an *ellipsoid*. This surface is symmetric with respect to all three coördinate planes and lies entirely within the limits $-a \leqq x \leqq a$, $-b \leqq y \leqq b$, $-c \leqq z \leqq c$.

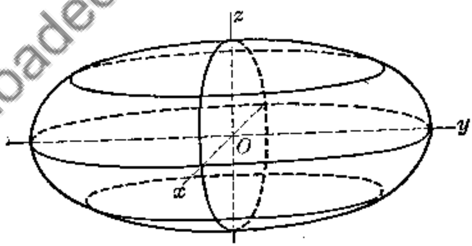

FIG. 86

The segments of length $2a$, $2b$, $2c$ cut off on the coördinate axes are the *axes* of the ellipsoid, the point O is the *center*.

113. The hyperboloid of one sheet. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (1)$$

represents a *hyperboloid of one sheet* (Fig. 87). The sections parallel to the yz- and zx-planes are hyperbolas; parallel to the xy-plane, ellipses. The intercepts on Ox are $\pm a$; on Oy, $\pm b$; on Oz, imaginary. The surface is a connected, open surface extending indefinitely in both directions along the z-axis.

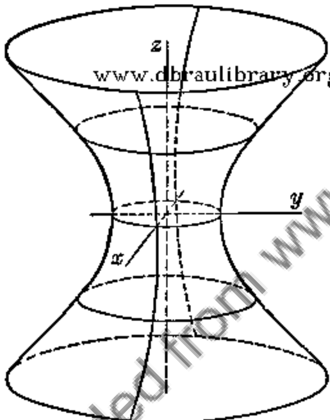

FIG. 87

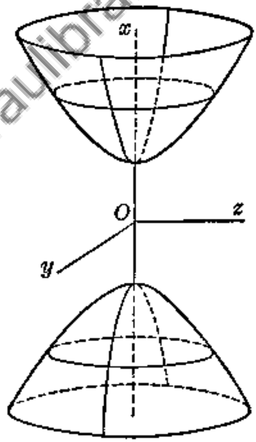

FIG. 88

114. The hyperboloid of two sheets. The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (1)$$

represents a *hyperboloid of two sheets* (Fig. 88). A study of the sections shows that the surface consists of two disconnected sheets, one in the region $x \geqq a$, the other in the region $x \leqq -a$, each opening out larger along Ox as x increases numerically. Note the unconventional arrangement of the axes in Fig. 88.

115. The elliptic paraboloid. The locus of the equation

(1) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

is called an *elliptic paraboloid*.

From equation (1) we find that the elliptic paraboloid has two planes of symmetry; it also has one line of symmetry, called the *axis* of the surface. The axis intersects the surface in a single point, called the *vertex*. The surface lies entirely on one side of the xy-plane, and extends to infinity along Oz.

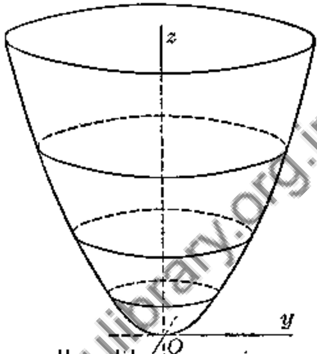

FIG. 89

116. The hyperbolic paraboloid. The surface

(1) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

is called a *hyperbolic paraboloid*. The sections $y = k$ are parabolas opening upward or downward, according to the sign of c; sections $x = k$ are parabolas opening in the opposite direction. It thus appears that the surface is "saddle-shaped," as shown in Fig. 90.

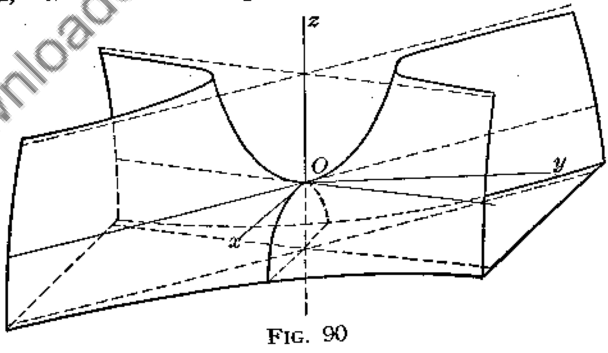

FIG. 90

EXERCISES

Classify and sketch the following surfaces.

1. $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{1} = 1.$

2. $\frac{x^2}{1} + \frac{y^2}{2} + \frac{z^2}{4} = 1.$

3. $\frac{x^2}{9} + \frac{y^2}{1} - \frac{z^2}{9} = 1.$

4. $\frac{z^2}{4} + \frac{x^2}{3} - \frac{y^2}{4} = 1.$

5. $\frac{x^2}{4} - \frac{y^2}{1} - \frac{z^2}{4} = 1.$

6. $\frac{z^2}{9} - \frac{x^2}{2} - \frac{y^2}{1} = 1.$

7. $3x^2 + 3y^2 + 4z^2 = 12.$

8. $x^2 - 4y^2 - 4z^2 = 1.$

9. $2x^2 + z^2 - 4y = 0.$

10. $x^2 + 2y^2 - 6z = 0.$

11. $x^2 - y^2 + 4z = 0.$

12. $x^2 - y^2 - z = 0.$

13. $x^2 + 2y^2 + 6z^2 = 6.$

14. $y^2 - 2z^2 - 2x^2 = 2.$

15. $2x^2 + 2z^2 = y.$

16. $4x^2 - 4z^2 = y.$

17. $4x + y^2 - 4z^2 = 0.$

18. $8x - y^2 - 4z^2 = 0.$

19. $x^2 + 4y^2 - 4z^2 + 9 = 0.$

20. $x^2 + y^2 + 4z^2 = 16.$

21. $8x + y^2 + 2z^2 = 0.$

22. $4x^2 + y + 4z^2 = 0.$

23. $x^2 + y^2 + z^2 = 2az.$

24. $x^2 - y^2 + 2z^2 + 2 = 0.$

25. Show that the ellipsoid (§ 112) is a prolate spheroid when the two shorter semi-axes are equal; an oblate spheroid when the two longer semi-axes are equal; a sphere when all three semi-axes are equal.

26. When do the surfaces of §§ 113–115 become surfaces of revolution?

27. Which of the surfaces in Exs. 1–24 are surfaces of revolution?

117. Cylinders. A *cylinder* is the surface described by a moving line which remains parallel to its original position and always intersects a fixed curve, called the *directing curve.* Thus the cylinder is completely covered by straight lines, called *generators*, all of which are parallel.

The section by any plane perpendicular to the generators is called a *right section;* it is obvious that all right sections, and in fact all parallel plane sections, are identical curves. If the right sections have centers, the line through these centers is the *axis* of the cylinder.

118. Equations in two variables: cylinders perpendicular to a coördinate plane. The argument employed in § 98 in no way depends on the fact that the base-curve is a straight line. But if, in Fig. 83, the line LM were curved, the surface $LMRS$ would be, not plane, but cylindrical. Thus the argument of that section serves to establish the very important

THEOREM: *An equation in two variables represents a cylinder whose generators are perpendicular to the plane of the two variables and whose directing curve is the curve represented by the given equation in that plane; and conversely.*

COROLLARY: *A cylinder is a quadric surface if and only if its right section is a conic.*

The truth of the corollary appears at once from the fact that, when the generators are perpendicular to a coördinate plane, the equation of the cylinder is identical with the equation (in the coördinate plane) of the directing curve.

A quadric cylinder is called *elliptic*, *parabolic*, or *hyperbolic*, according to the nature of its right section.

119. Cones. A *cone* is the surface generated by a moving line that always passes through a fixed point, called the *vertex*, and intersects a fixed *directing curve*. Thus the surface is completely covered by straight lines, or *generators*, all passing through a fixed point. Like the cylinder, a cone may or may not be a quadric surface.

120. The elliptic cone. The locus of the equation

$$\text{(1)} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

is a *quadric cone*. This surface evidently has three planes of symmetry, and a center of symmetry, called the *vertex*. A study of the sections shows that the surface consists of two open sheets extending indefinitely along the z-axis (Fig. 91).

By suitable coördinate transformations (analogous to those of §§ 55–56), the equation of every quadric cone can be reduced to the form (1). Hence there is only a single species of quadric cone; from this one surface every kind of conic can be cut. Since the surface is most clearly visualized by means of its elliptic sections, it is usually called the *elliptic cone*.

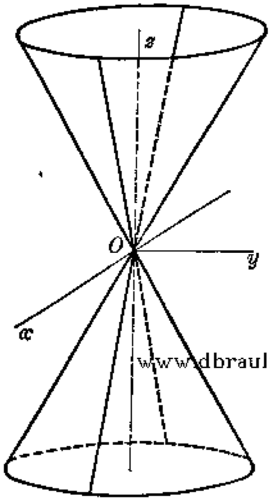

FIG. 91

The line through the centers of the elliptic sections is called the *axis* of the cone, and sections by planes perpendicular to the axis are *right sections*.

When $a = b$, the surface is the *circular cone*.

EXERCISES

Draw the following cylinders.

1. $z^2 = 4ay$. **2.** $x^2 - 4z^2 = 16$.

3. $x^2 + y^2 = 4ay$. **4.** $2xy = a^2$.

5. $z = x^2 - 2x$. **6.** $2y^2 + z^2 = 4y$.

7. $y^2 - z^2 = 2y$. **8.** $x^2 - 2y = 2$.

Draw the following cones.

9. $4x^2 - 9y^2 + 36z^2 = 0$. **10.** $4x^2 - y^2 - z^2 = 0$.

11. $x^2 + y^2 = 9z^2$. **12.** $3x^2 + y^2 - 3z^2 = 0$.

13. $9x^2 - 4y^2 = 9z^2$. **14.** $2x^2 - 3y^2 + 4z^2 = 0$.

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